

Counting Leaders of Labeled Rooted Trees

有根标号树中 Leader 的计数问题

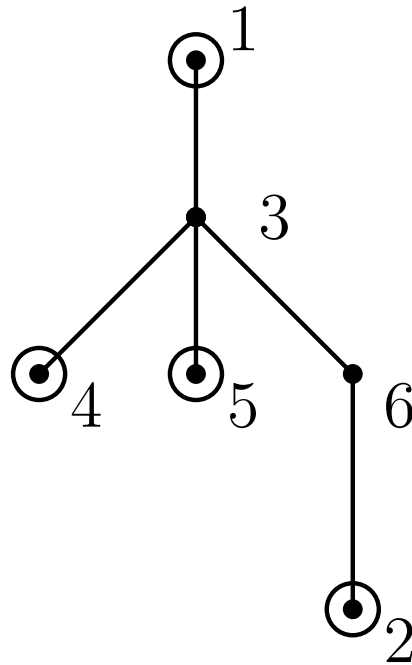
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Leaders

A vertex v is called a **leader** (or **proper vertex** by Seo) if v is **minimal** among its descendants.



Backgrounds

- The enumeration of leaders generalize Cayley's formula (Gessel and Seo):

$$\sum_T u^{\text{lead}(T)} c^{\text{deg}_T(1)} = cu^2 \prod_{i=1}^{n-2} (i + (n-1-i)u + cu), \quad (1)$$

where T ranges over all trees on $[n]$ with root 1.

Setting $u = c = 1$, we immediately get n^{n-2} , which is the number of all trees on $[n]$ with root 1.

Backgrounds

- Another motivation comes from Postnikov's hook length formula:

$$(n + 1)^{n-1} = \frac{n!}{2^n} \sum_T \prod_v \left(1 + \frac{1}{h(v)} \right),$$

where T ranges over all binary trees on $[n]$, v ranges over all vertices of T , and $h(v)$ is the number of descendants of v , including v .

Gessel and Seo, Chen and Yang, Du and Liu provides different proofs.

Previous Work

- Gessel and Seo proved (1) and provide similar formulas for
 - forests,
 - k -ary trees,
 - k -colored ordered trees and
 - k -colored ordered forests.
- Seo and Shin gave a combinatorial proof using a variation of the Prüfer code which they called it the reverse Prüfer code.

Our Approach

Trees on $[n]$



Sequences $a_1 a_2 \cdots a_{n-1}$
with $a_i \in \{1, 2, \dots, n\}$

Leaders

=

$1 + \# \{i : a_i \leq i\}$

$\deg_T(1)$

=

occurrences of 1's

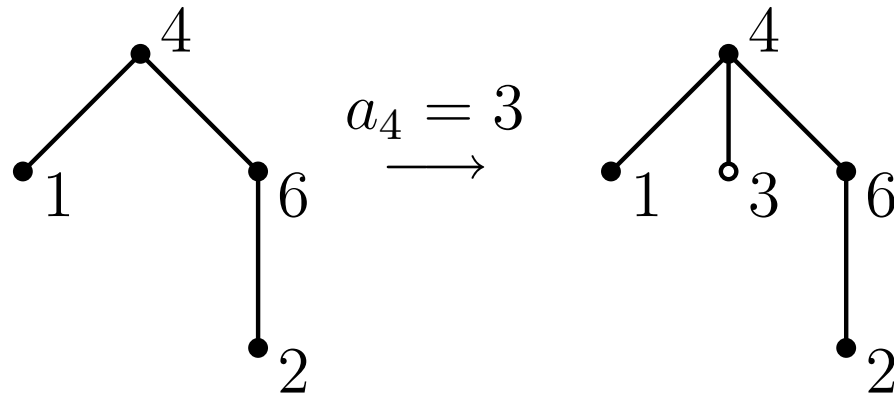
Sequences \Rightarrow Trees

The tree is constructed recursively.

- Initially, we have a tree which consists of only one vertex, the root 1.

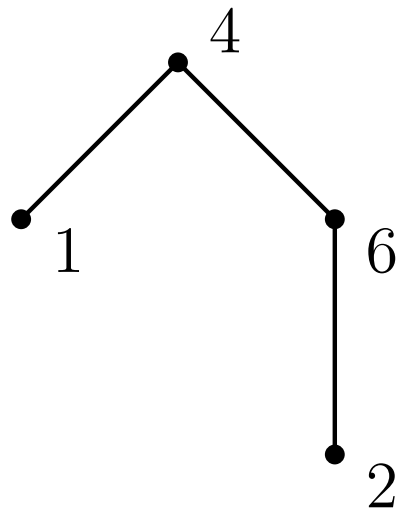
$$T_0 = \bullet^1$$

- To construct T_k from T_{k-1} , we encounter two cases:
 - $a_k \leq k$



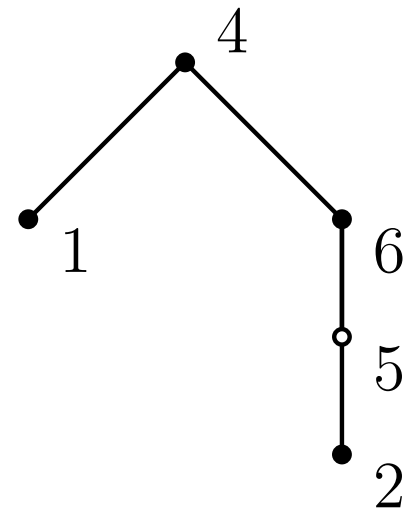
Sequences \Rightarrow Trees

- $a_k > k$

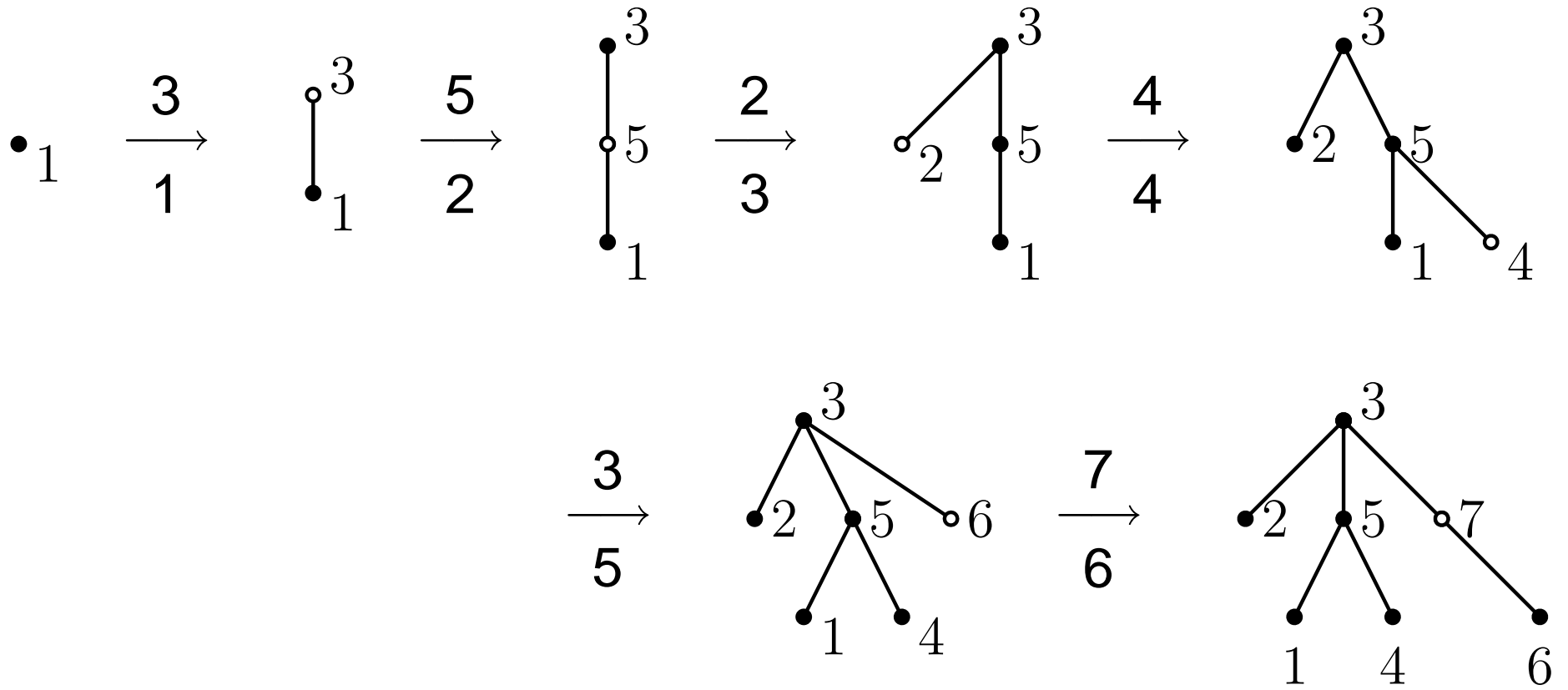


$$a_4 = 6$$

→



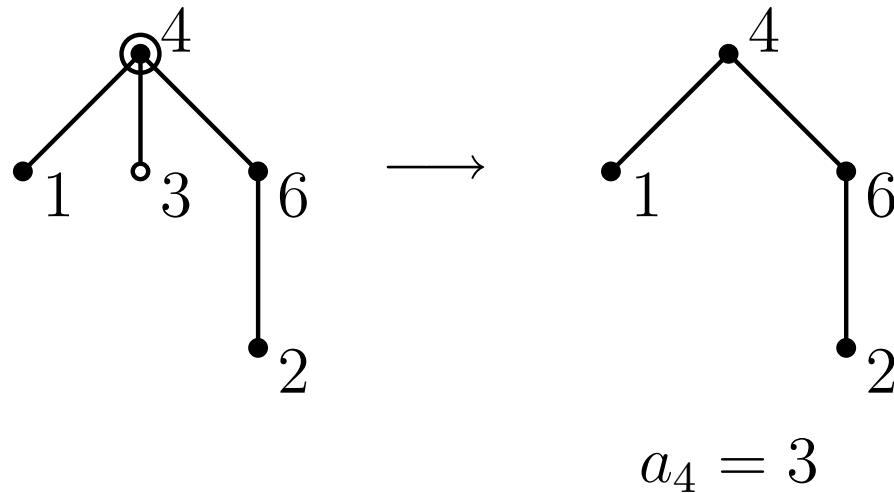
Example



Trees \Rightarrow Sequences

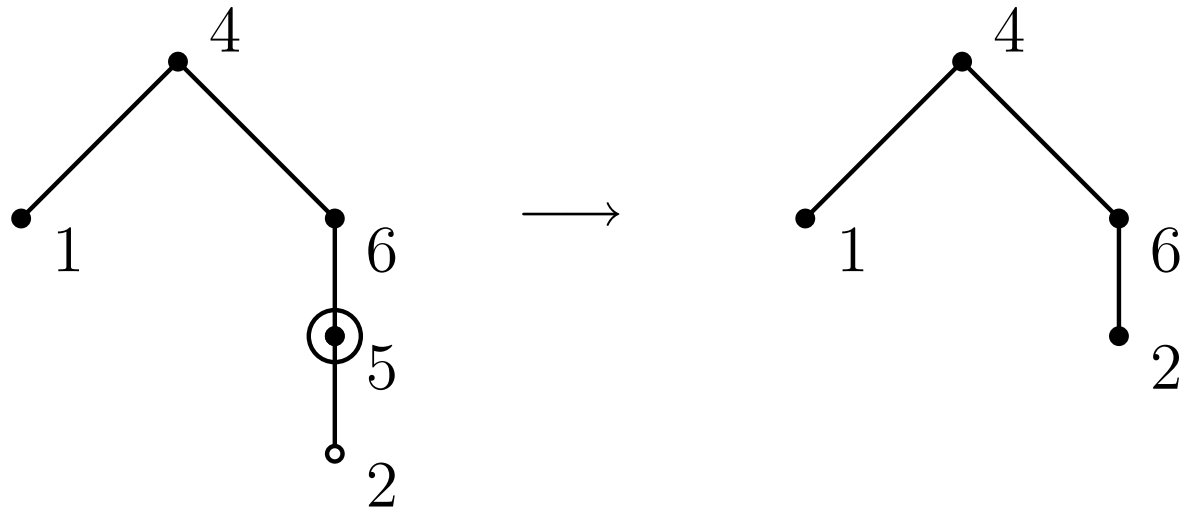
Conversely, we can recover the sequence from the tree.

- Initially, we have $T_{n-1} = T$.
- To construct T_{k-1} from T_k , we consider the **largest leaf** j and its parent j' .
 - $j' < j$ or j' has more than one children.



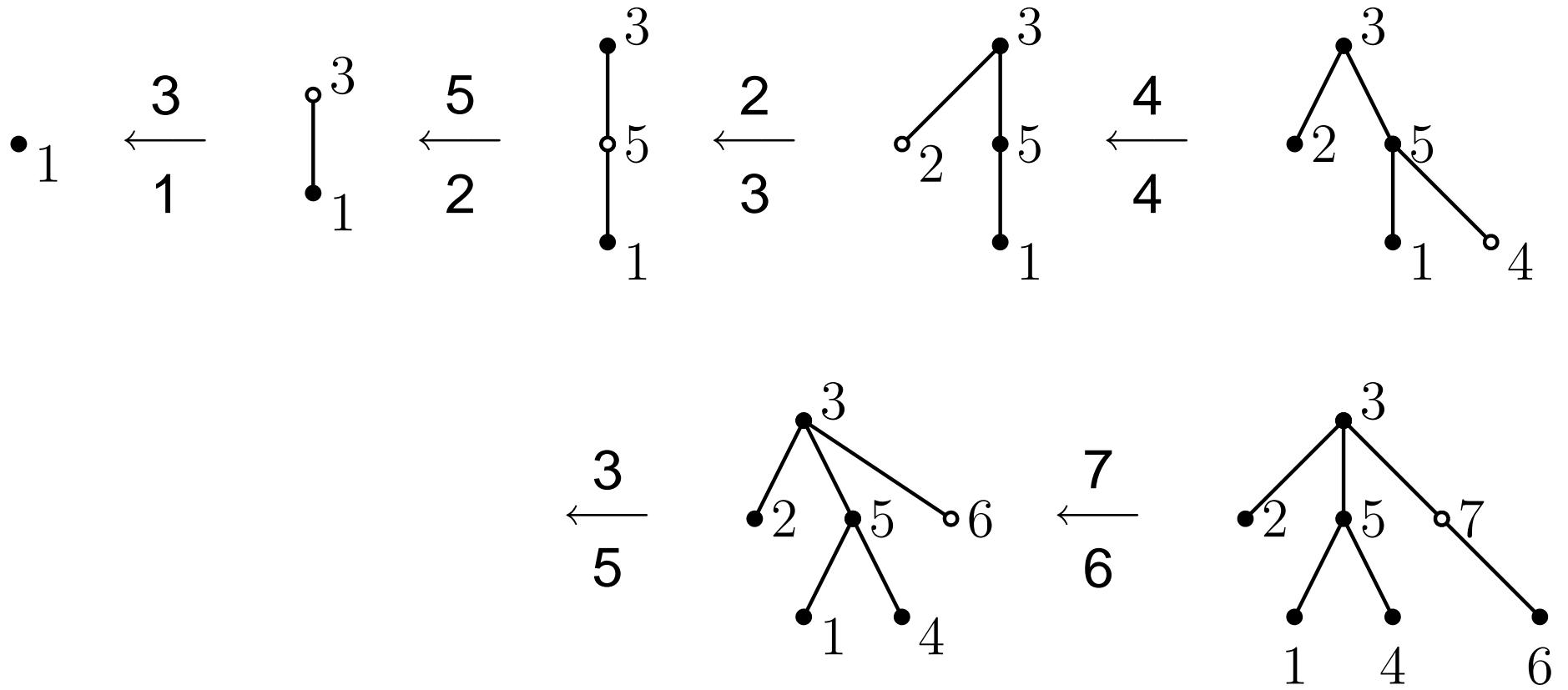
Trees \Rightarrow Sequences

- $j' > j$ and j is the only child of j' .



$$a_4 = 4 + 2 = 6$$

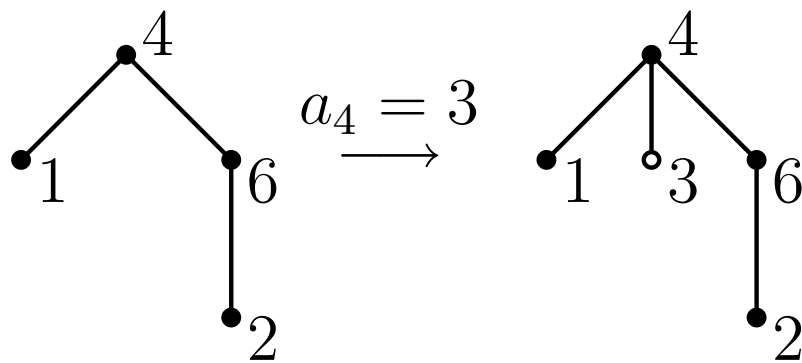
Example



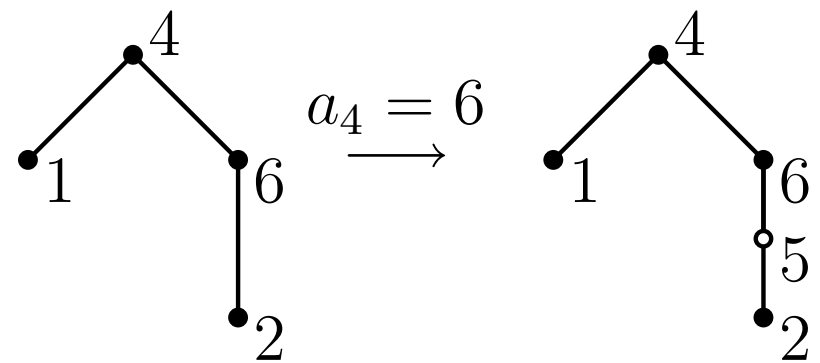
LF property

The trees T_k have a nice property, which we call it the **leaf filling** (LF for short) property:

For each $0 \leq i \leq n - 1$, if j is a leaf of T_i , then $[j] \subseteq S_i$.



Case (a)



Case (b)

Number of leaders and $\deg_T(1)$

$$\# \text{ Leaders in } T_k = 1 + \# \{i: a_i \leq i, 1 \leq i \leq k\}$$

$$\deg_T(1) = \# \text{ occurrences of 1's}$$

Corollary 1 *We have*

$$\sum_T u^{\text{lead}(T)} x_1^{\text{deg}_T(1)} x_2^{\text{deg}_T(2)} = x_1 u^2 \prod_{i=1}^{n-2} (x_1 u + x_2 u + (i-1)u + (n-1-i)),$$

where T ranges over all trees on $[n]$ with root 1.

Generalization

Slight modifications of the bijection can prove the formulas for

- forests

$$\sum_F a^{n-\text{lead}(F)} b^{\text{lead}(F)-\text{tree}(F)} c^{\text{tree}(F)} = c \prod_{i=1}^{n-1} (ia + (n-i)b + c). \quad (2)$$

- k -ary trees

$$\sum_U u^{\text{lead}(U)} = u \prod_{i=1}^{n-1} ((n-i)k + (ik - i + 1)u). \quad (3)$$

Generalization

- k -colored ordered trees

$$\sum_T u^{\text{lead}(T)} = u \prod_{i=1}^{n-1} ((n-i)k + (ik + i - 1)u). \quad (4)$$

- k -colored ordered forests.

$$\begin{aligned} \sum_F u^{n-\text{lead}(F)} v^{\text{lead}(F)-\text{comp}(F)} w^{\text{comp}(F)} \\ = w \prod_{i=1}^{n-1} (iku + (n-i)(k+1)v + w). \end{aligned} \quad (5)$$

Thanks for attending