Integer Exact Network Synthesis Problem

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Introduction

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- 2. Main algorithm

  Subroutine 1–Algorithm Tree Finding

  Subroutine 2–Algorithm Cut-tree Realization

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- Let $R = (r_{ij})_{n\times n}$ be a symmetric, non-negative matrix of minimum flow requirements between all pairs of distinct nodes in $V = \{1, 2, \ldots, n\}$, where $r_{ii} = 0\ i \in \{1, \ldots, n\}$.

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- A realization of R: a network $G = [V, E, u]$ on node set V with edge set E, and non-negative edge capacities $\{u_e : e \in E\}$ if and only if for every pair $\{i, j\}$ of distinct nodes in V, the value of maximum flow between i and j in G is at least r_{ij}.

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- If R has an exact realization, then it is exactly realizable.

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- **Four Problems.**

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- **NS** Problem–Network Synthesis Problem

  ENS Problem–Exact Network Synthesis Problem

  INS Problem–Integer Network Synthesis Problem

  IENS Problem–Integer Exact Network Synthesis Problem

  Each of the four problems is to construct a required realization of R with minimum sum of edge capacities.

Introduction

- Let f_{ij} denote the maximum flow value or minimum cut value for each pair nodes i, j.

NS Problem

$$\min \sum_{e \in E} u_e$$

s. t. $$f_{ij} \geq r_{ij}$$

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- **NS** problem : Gomory-Hu(1961), Gusfield(1983), Tardos (1986), Talluri(1996), solved the problem in strongly polynomial time, respectively.

  ENS problem. A modification of the Gomory-Hu algorithm produces an optimal solution for **ENS** problem whenever it has a feasible solution.

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- **INS** Problem

  min $\sum_{e\in E} u_e$

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- **INS** problem. Chou and Frank(1970), Sridhar and Chandrasekaran(1992), presented combinatorial algorithms of computational complexity $O(n^2)$.

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- **INS** problem. Chou and Frank(1970), Sridhar and Chandrasekaran(1992), presented combinatorial algorithms of computational complexity $O(n^2)$.
- **IENS** Problem. Chou, Frank claim to give an algorithm for the problem. However, Schrijver gives a counter-example to their claim(Schrijver, A. 2003. *Combinatorial optimization: Polyhedra and efficiency*, Algorithms and Combinatorics 24, Springer-Verlag, New York.).

Introduction: An counter-example

- **An counter-example.** $R = \begin{pmatrix} 0 & 5 & 3 & 3 & 3 & 3 \\ 5 & 0 & 3 & 3 & 3 & 3 \\ 3 & 3 & 0 & 5 & 5 & 3 \\ 3 & 3 & 5 & 0 & 5 & 3 \\ 3 & 3 & 5 & 5 & 0 & 3 \\ 3 & 3 & 3 & 3 & 3 & 0 \end{pmatrix}$

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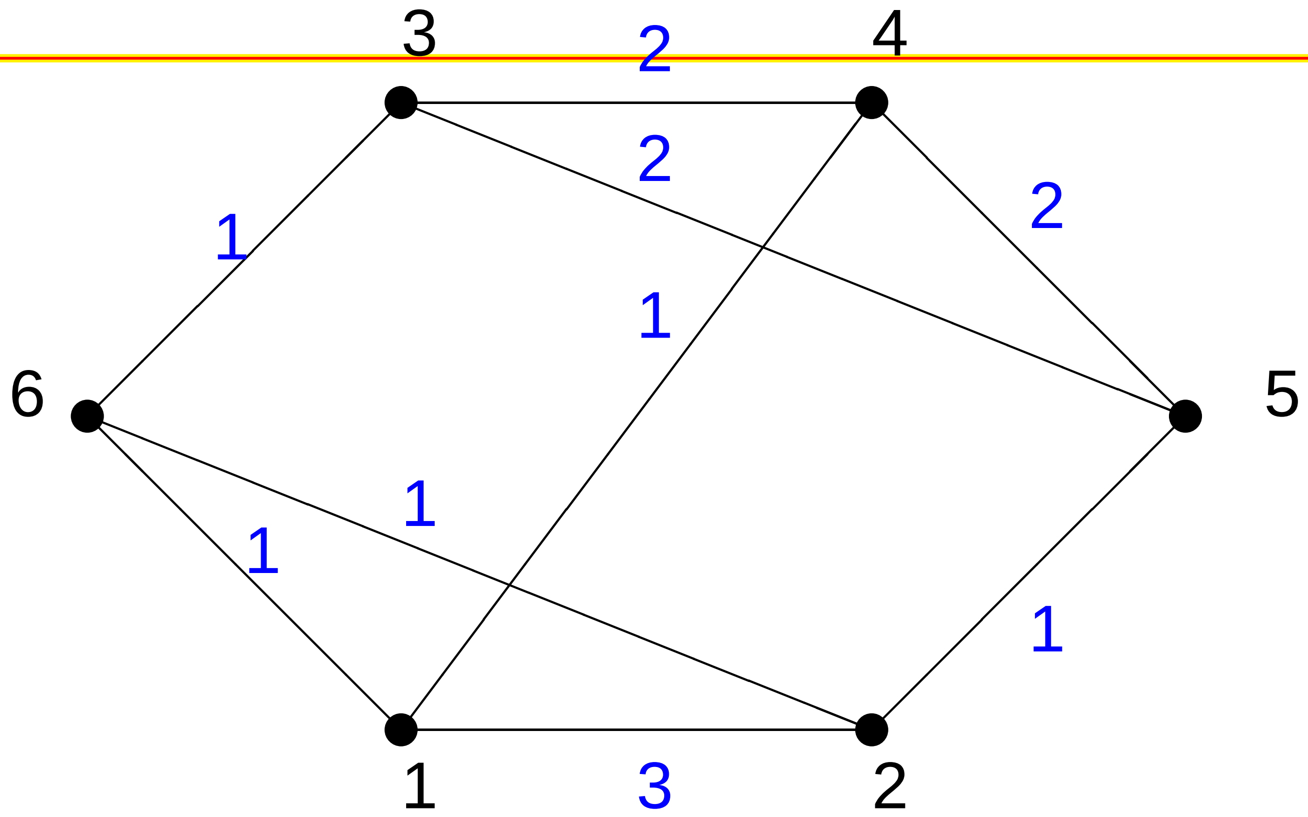
- Chou-Frank Algorithm gives the total capacities $\sum_{e \in E} u_e = 15$, but there is an exact realization with $\sum_{e \in E} u_e = 14$.

Introduction: known results



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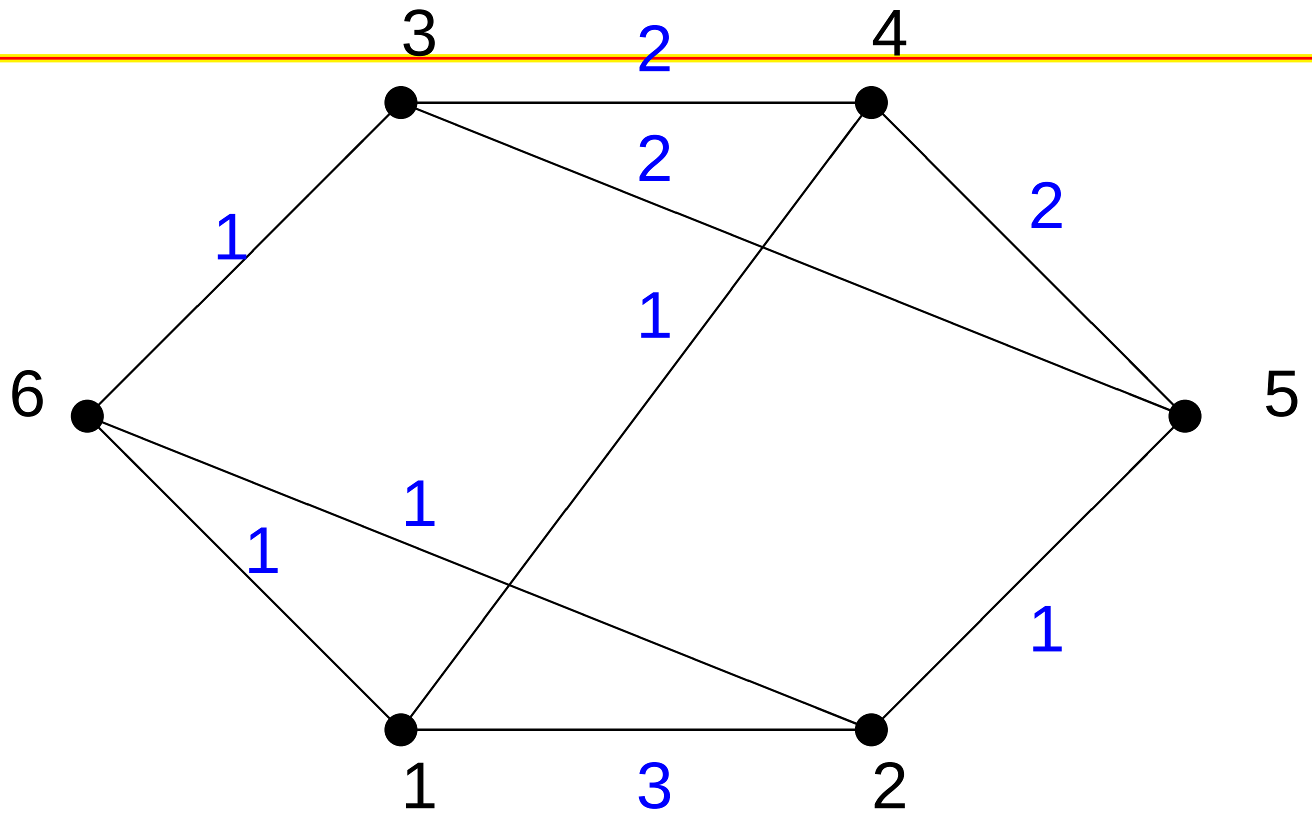

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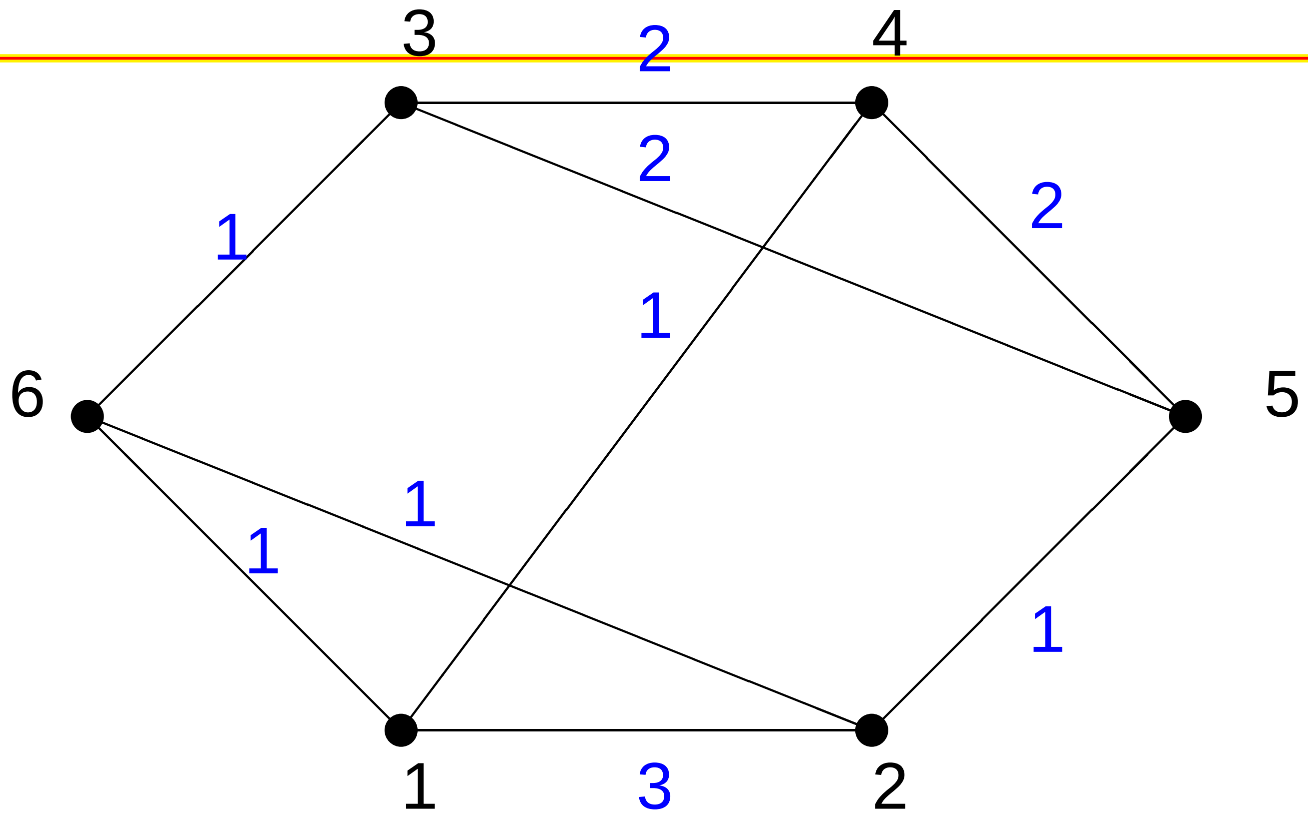
- A realization $G = [V, E, u]$ of R with $\sum_{e \in E} u_e = 14$.

Introduction: known results



- A realization $G = [V, E, u]$ of R with $\sum_{e \in E} u_e = 14$.
- We give an $O(n^2)$ combinatorial algorithm for the **IENS** problem.

2. Main algorithm-definition and theorems

- Definition 2.1 Two networks $G^1 = [V, E^1, u^1]$ and $G^2 = [V, E^2, u^2]$ on the same node set V and with edge capacities $\{u_e^1 : e \in E^1\}$ and $\{u_e^2 : e \in E^2\}$ are said to be flow-equivalent if for any pair $\{i, j\}$ of distinct nodes in V, the maximum flow values between i and j in the two networks are the same.

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- **Theorem 2.1**[9, 21]There exists an $O(n^2)$ algorithm to test if a given matrix is exactly realizable.

2. Main algorithm-definition and theorems

- **Theorem 2.2**[9] For a symmetric, $n \times n$, non-negative matrix R, let G^R be a complete network on node set $V = \{1, 2, \ldots, n\}$ with capacity of each edge (i, j) equal to r_{ij}. Then
  (I) Every maximum weight spanning tree of G^R is a realization of R.
  (II) The following three statements are equivalent.
  (IIa) R is exactly realizable.
  (IIb) $r_{ij} \geq \min\{r_{ik}, r_{kj}\}$ for all distinct $i, j, k \in V$.
  (IIc) Every maximum weight spanning tree $T = [V, E]$ of G^R with edge capacities
  $u_e = r_{ij} \; \forall e = (i, j) \in E$ is an exact realization of R.

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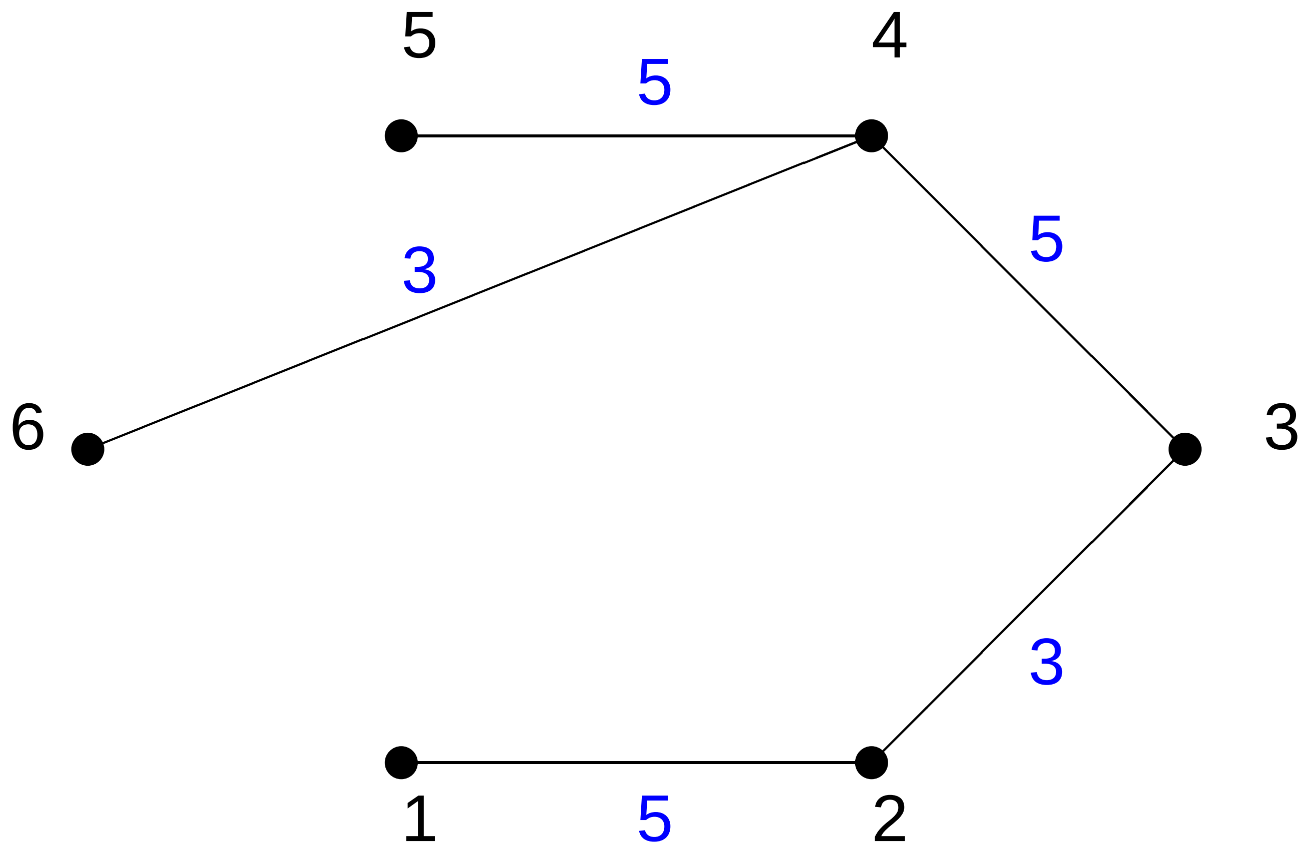
- Input. A symmetric, integer, non-negative matrix $R = (r_{ij})_{n \times n}$.
- Output. A network $G^* = [V, E^*, u^*]$, that is an optimal solution to the instance of the **IENS** problem.
- Step 0. Find a maximum weight spanning tree $T = [V, E, u]$ in G^R, (the complete graph on node set V with edge weight $u_e = r_{ij} \; \forall \;\; e = (i, j), i \neq j$.

2. Main algorithm-an example

- **An example.** $R = \begin{pmatrix} 0 & 5 & 3 & 3 & 3 & 3 \\ 5 & 0 & 3 & 3 & 3 & 3 \\ 3 & 3 & 0 & 5 & 5 & 3 \\ 3 & 3 & 5 & 0 & 5 & 3 \\ 3 & 3 & 5 & 5 & 0 & 3 \\ 3 & 3 & 3 & 3 & 3 & 0 \end{pmatrix}$

3. Main algorithm-an example



- A maximum weight spanning tree $T = [V, E, u]$ in G^R.

2. Main algorithm

- If T does not exactly realizes R, then conclude that "R is not exactly realizable" and stop.
  Otherwise, let $\bar{E} = \{e : e \in E, u_e \leq 1\}$, $F = [V, E - \bar{E}, u]$.
  Let $T^1, T^2, \cdots, T^k$ be the connected components of F with node sets $V^1, \ldots, V^k$, respectively. Set $i = 1$.

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- Step 1. Find an optimal solution $T^{*i} = [V^i, E^i, u^i]$ with input T^i, using **Algorithm Tree Finding**.

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- Step 1. Find an optimal solution $T^{*i} = [V^i, E^i, u^i]$ with input T^i, using **Algorithm Tree Finding**.
- Step 2. Find optimal solution $G^{*i} = [V^i, E^{*i}, u^{*i}]$ with input T^{*i}, using **Algorithm Cut-tree Realization**.

2. Main algorithm

- **Step 3.** If $i < k$, then increment $i = i + 1$ and go to Step 1. Else, construct network $G^* = [V, E^*, u^*]$ with $E^* = (\cup_{i=1}^{k} E^{*i}) \cup \bar{E}$ and

$$u_e^* = \begin{cases} u_e^{*i} & \text{if } e \in E^{*i} \text{ for some } i \in \{1, 2, \cdots, k\} \\ u_e & \text{if } e \in \bar{E} \end{cases}$$

Output the network G^* and stop.

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- **Theorem 3.1** Main Algorithm produces an optimal network $G^* = [V, E^*, u^*]$ with optimal value $u^*[V] = \frac{\sum_{i \in V} \pi_i + |V_1^e| + |V_2^o|}{2}$ to the **IENS** problem in $O(n^2)$ time.

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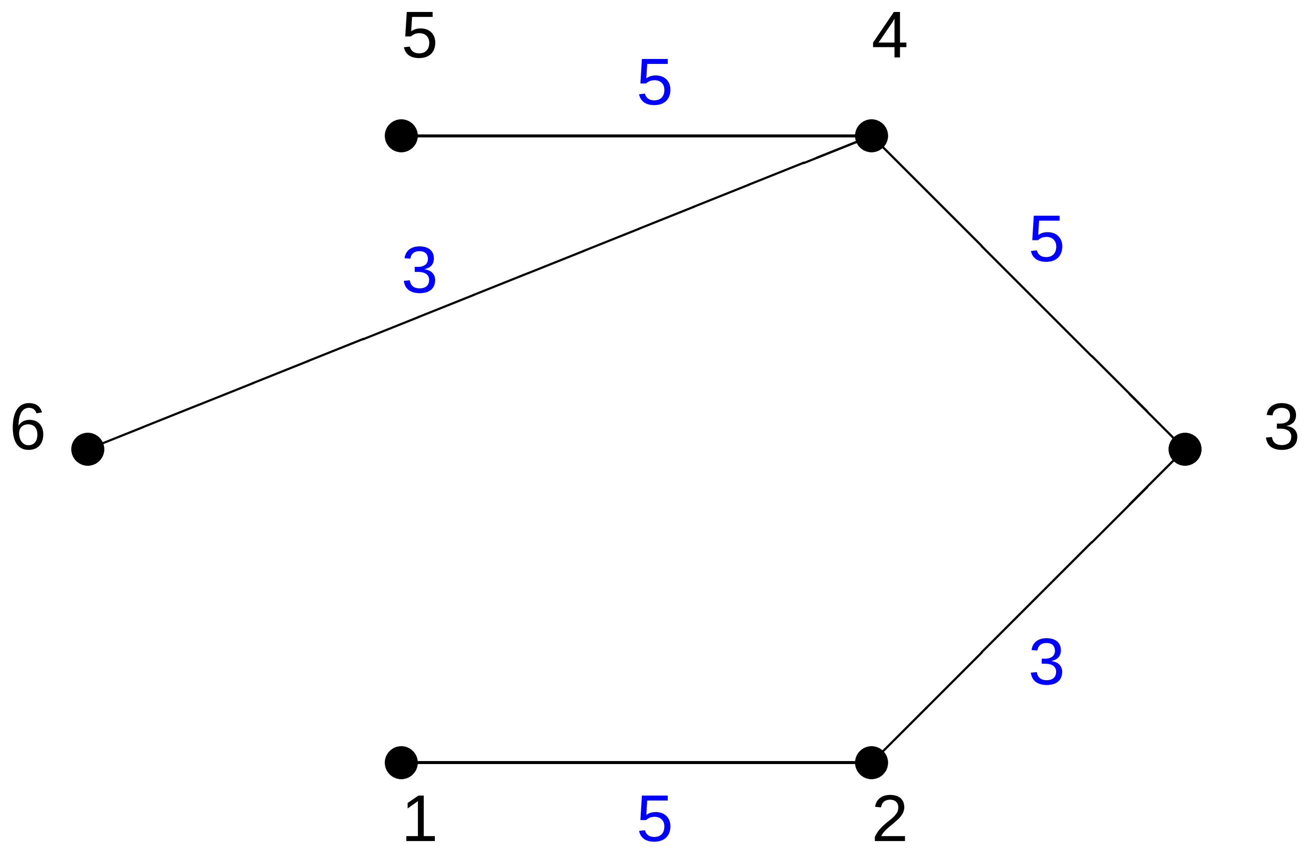

- A maximum weight spanning tree T

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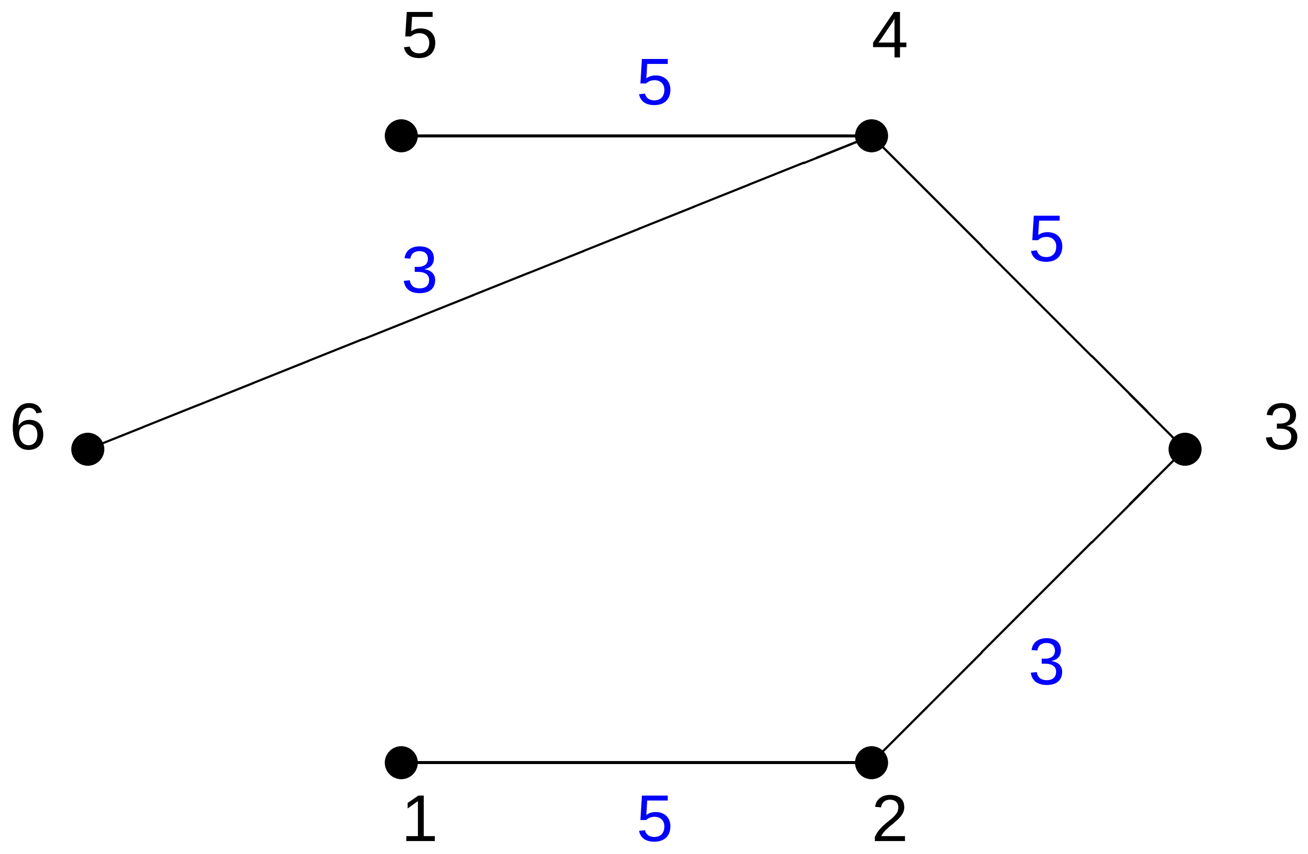
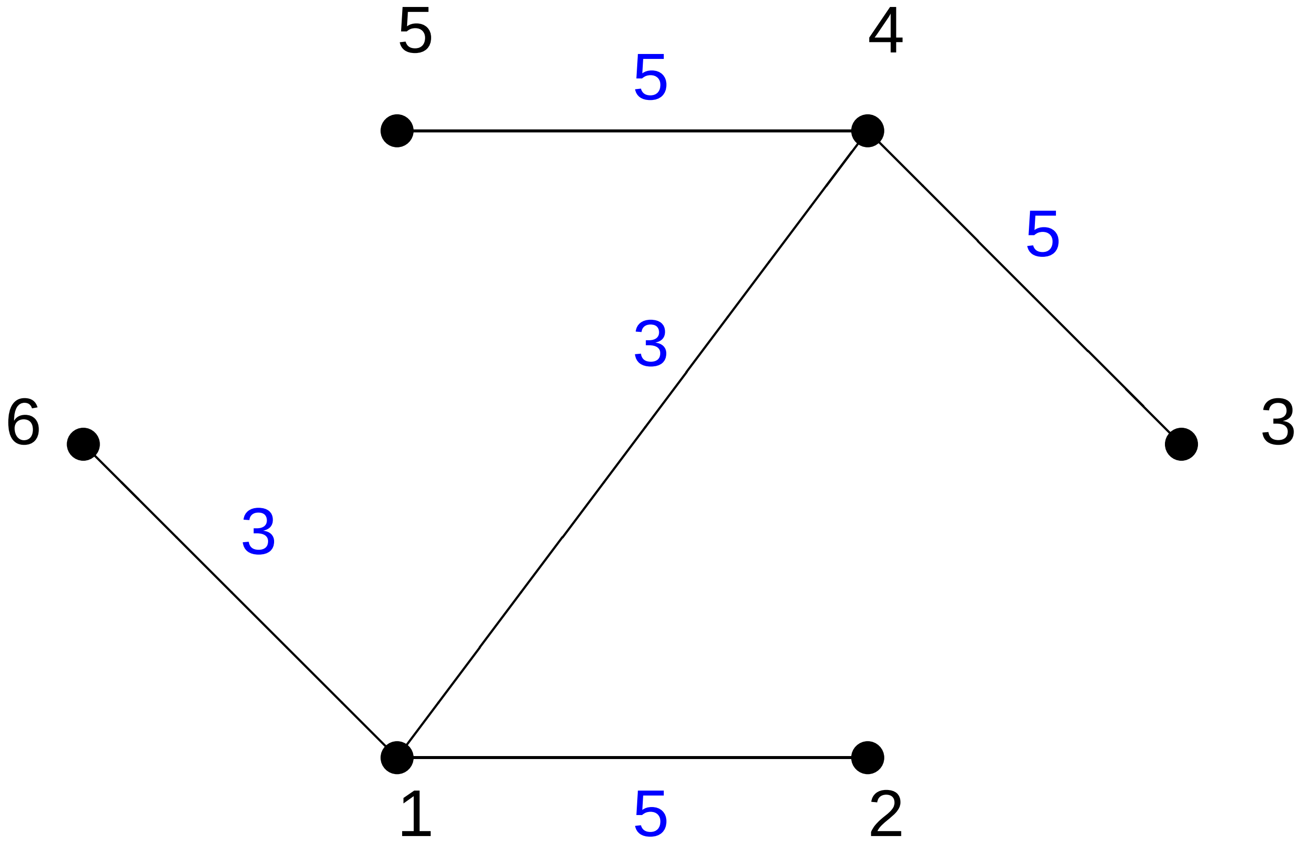


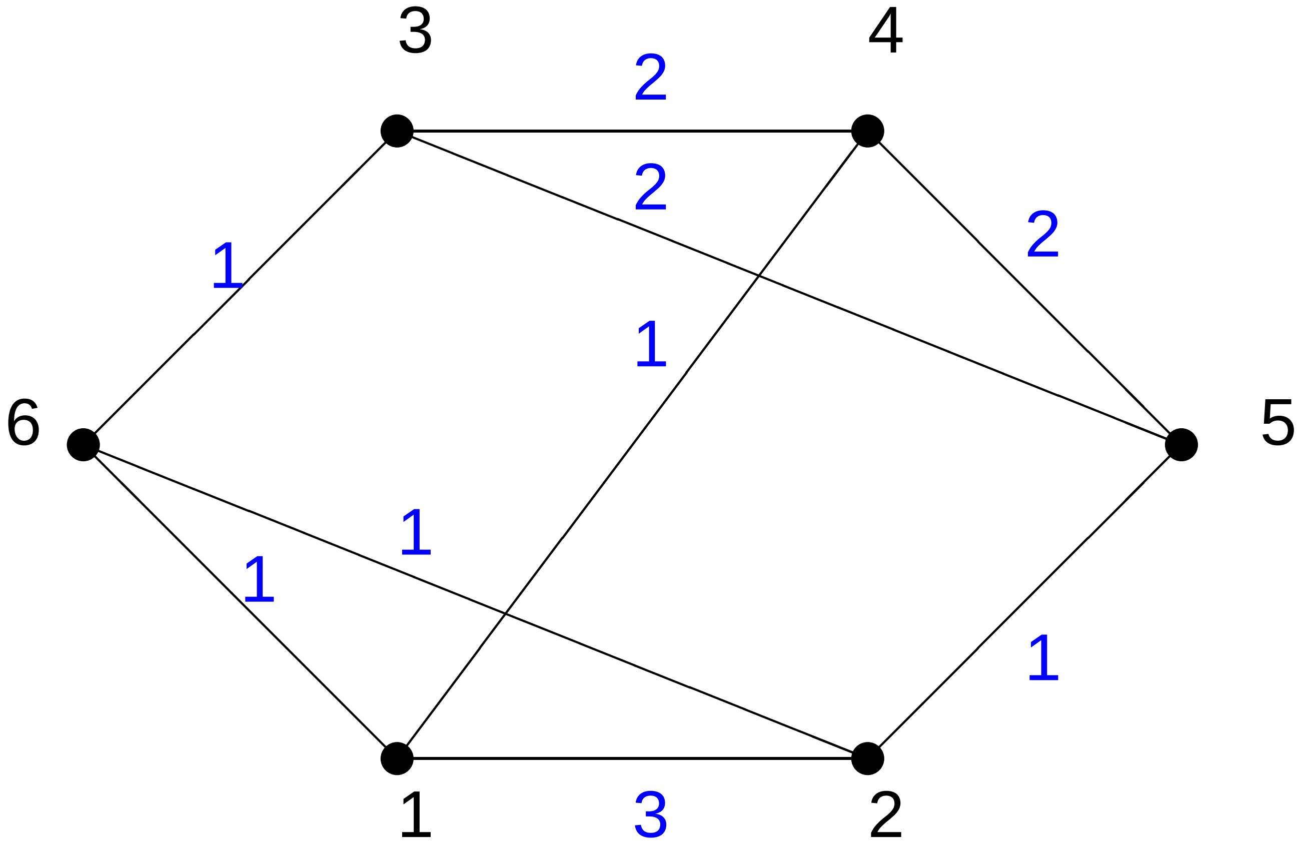
- A maximum weight spanning tree T
- R is exactly realizable.

3. An example



- An optimal tree T^* that is flow equivalent to the maximum weight spanning tree T, using **Algorithm Tree Finding.**

3. An example



- An optimal exactly realization G of R, constructed by **Algorithm Cut-tree Realization**

Thanks

Thank you for coming!

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