Abstract

Symmetry of graphs has been extensively studied over the past 50 years by using automorphisms of graphs and group theory which have played an important role in graph, and promising and interesting results have been obtained[2,20,21]. We introduced generalized symmetry of graphs and investigated it by using endomorphisms of graphs and semigroup theory. In this paper, I will survey some results we have achieved in recent years. The paper consists of the following sections.

1. Introduction
2. End-regular graphs
3. Weakly transitive graphs
4. Cayley graphs of semigroups
5. Unretractive graphs

Keywords: Graph, endomorphism, monoid, generalized symmetry.
1 Introduction

For graphs and semigroups, there are many aspects which are very interesting and worth investigating. In this paper, we mainly focus on studying generalized symmetry of graphs using endomorphism semigroups of graphs and semigroup theory.

Section 2 is devoted to End-regular graphs. The definition of End-regular graph is given by Fan [3,4]. The initial aim is to answer the open problem raised by Knauer [37]. It is interesting from the characterization of End-regular graphs (Theorem 2.2) we know that these graphs do have some symmetry although the definition is actually given in the sense of algebraic property of semigroups. In End-regular graphs, any image subgraph is a retract, and there exists another retract such that these two retracts are isomorphic.

Weakly symmetric graphs are introduced and studied in Section 3. These graphs are natural generalization of symmetric graphs. It is well known that almost all graphs are asymmetric. So, the study of weakly symmetric graphs is important and necessary. The structure properties, classifications and combinatorial properties of these graphs remain to be given. The study of this area now is just a start.

Cayley graphs of groups play important role in symmetric graphs. Therefore, Cayley graphs of semigroups are also important part of generalized symmetric graphs. This topic will receive increasing attention in the near future. Some results for Cayley graphs of semigroups are presented in Section 4.

There are various unretractive graphs, among them two main unretractive graphs are E-A and E-S. E-A unretractive graphs, called cores in [20,21], are key to graph homomorphisms and symmetry. Our results in Section 5 show that E-S unretractive graphs can be constructed from E-A unretractive graphs.

We list some other results about graphs and their endomorphism monoids in final section.

Throughout this paper graphs are finite, simple, undirected, and connected, unless specified otherwise. Let \( X \) be a graph with vertex set \( V(X) \) and edge set \( E(X) \). We write \( \{x_1, x_2\} \in E(X) \) if the vertices \( x_1 \) and \( x_2 \) are adjacent in \( X \). For two graphs \( X \) and \( Y \), a mapping \( f : V(X) \rightarrow V(Y) \) is said to be a homomorphism from \( X \) to \( Y \) if \( f \) preserves adjacency of vertices, i.e., \( \{x_1, x_2\} \in E(X) \) implies \( \{f(x_1), f(x_2)\} \in E(Y) \) for any \( x_1, x_2 \in V(X) \); a strong homomorphism from \( X \) to \( Y \) if \( \{x_1, x_2\} \in E(X) \) if and only if \( \{f(x_1), f(x_2)\} \in E(Y) \) for any \( x_1, x_2 \in V(X) \); an isomorphism if \( f \) is bijective and \( f \) is a strong homomorphism from \( X \) to \( Y \). If \( X = Y \), then the above mappings are called endomorphisms, strong endomorphisms and auto-
morphisms respectively. Let $\text{End } X$ be the set of all endomorphisms of $X$, $\text{SEnd } X$ the set of all strong endomorphisms of $X$ and $\text{Aut } X$ the set of all automorphisms of $X$. Then $\text{End } X$ forms a monoid under composition, and is called the endomorphism monoid of $X$. Similarly, we have the strong endomorphism monoid $\text{SEnd } X$ and the automorphism group $\text{Aut } X$. Clearly, $\text{Aut } X \subseteq \text{SEnd } X \subseteq \text{End } X$.

For graph terminology and notation, please refer to [2,20].

2 End-regular graphs

Open question (Ulrich Knauer [37]): Which graphs have a regular endomorphism monoid? We answered it for bipartite graphs [3,4].

Definition 2.1 A graph $X$ is said to be End-regular if its endomorphism monoid $\text{End } X$ is regular. That is, for any $f \in \text{End } X$, there exists $g \in \text{End } X$ such that $fgf = f$.

Let $f$ be an endomorphism of $X$. Denote by $I(f)$ the image subgraph of $X$ under $f$, which is the induced subgraph with vertex set $f(V(X))$. An induced subgraph $A$ is a retract of a graph $X$ if there is an endomorphism $f$ such that $A = I(f)$ and $f(a) = a$ for each $a \in A$.

Theorem 2.2 Let $X$ be a graph, then $X$ is End-regular if and only if for any endomorphism $f$, the image subgraph $I(f)$ is a retract of $X$ and there exists an induced subgraph (or a retract) of $A$ of $X$ such that $f|_{V(A)}$ is an isomorphism from $A$ onto $I(f)$.

Although End-regular graph is defined in the semigroup sense, Theorem 2.2 gives a characterization of End-regular graphs using properties of graphs. Also, the theorem shows that there are a lot of isomorphic retracts in these graphs. Put in another way, End-regular graph do have some kind of symmetry. Using the structure properties of bipartite graphs and their retracts, we give End-regular bipartite graphs in the following.

Theorem 2.3 Let $X$ be a bipartite graph. Then $X$ is End-regular if and only if $X$ is one of the following graphs.
1. a complete bipartite graph $K_{n,n}, n, m \geq 1$;
2. a double star $T(s, t), r, s \geq 1$;
3. a cycle $C_6$ of length 6;
4. a path $P_5$ of length 4, or a cycle $C_8$ of length 8;
5. a disconnected graph $nK_1, (n - 1)K_1 \cup K_2$, or $nK_2, n \geq 2$. 
Notice that a regular endomorphism monoid of a graph was characterized by means of idempotents in (W.M.Li,[34]), and E.Wilkeit independently obtained results for connected bipartite graphs with a regular endomorphism monoid [41]. S.H.Fan [10,18] and W.M.Li [36,35] also studied End-regularities for split graphs and circulant graphs.

Problem 2.4 Determine End-regular non-bipartite graphs.

S.H.Fan considered some strong End-regularity and weak End-regularity of graphs. See [8] and [11] for details. For the regularity of a strong endomorphism monoid $SEnd X$ of a graph $X$, U.Knauer [31] and W.Li [33] proved that $SEnd X$ is regular if $X$ is finite. We obtained the following results [12,14].

Theorem 2.5 Let $X$ be an infinite graph, $U$ be its canonical strong factor graph. Then the following statements are equivalent.

(1) $SEnd X$ is a regular monoid.
(2) Any strong image subgraph of $X$ is a strong retract.
(3) $U$ is S-A unretractive.
(4) $U$ contains no proper subgraph which is isomorphic to $U$.

3 Weakly transitive graphs

In this section we introduce and study weakly transitive graphs (or weakly symmetric graphs).

Definition 3.1 [3,16,19] A graph $X$ is said to be weakly vertex transitive if its endomorphism monoid $End X$ acts transitively on the vertex set. That is, for any two vertices $x, y \in V(X)$, there exists $f \in End X$ such that $f(x) = y$.

A graph $X$ is a core if any endomorphism of $X$ is an automorphism or, equivalently, if its endomorphism monoid equals its automorphism group. An induced subgraph $A$ of $X$ is a core of $X$ if $A$ is a core and $A$ is an endomorphism image subgraph of $X$. So, a core of $X$ is an endomorphism image subgraph with the smallest vertices. Every graph has a core, and all its cores are isomorphic. See [20] for details. The following theorem gives the close relation between weakly vertex transitive graphs and vertex transitive graphs[19].

Theorem 3.2 A graph $X$ is weakly vertex transitive if and only if its core $A$ is vertex transitive and any vertex of $X$ lies in some induced subgraph of $X$ isomorphic to $A$.

Problem 3.3 How to construct weakly vertex transitive graphs from vertex transitive graphs?
Analogous definitions and results are obtained for weakly edge transitive graphs and weakly arc transitive graphs [3,19,43].

**Theorem 3.4** Bipartite graphs are weakly vertex transitive, weakly edge transitive and weakly arc transitive.

**Definition 3.5** A graph \( X \) is said to be weakly \( s \)-arc transitive if its endomorphism monoid \( \text{End} \ X \) acts transitively on the \( s \)-arc set. That is, for any two \( s \)-arcs \( \alpha = (x_0, \ldots, x_s) \), \( \beta = (y_0, \ldots, y_s) \), there exists \( f \in \text{End} \ X \) such that \( f \) maps \( \alpha \) to \( \beta \), i.e., \( f(x_i) = y_i, 0 \leq i \leq s \).

We have obtained the following results [43]. (1) A tree with diameter \( d \) is weakly \( s \)-arc transitive for all \( 0 \leq s \leq d \). (2) If a graph with girth \( g \geq 3 \) is weakly \( s \)-arc transitive, then \( s \leq (g + 2)/2 \). Moreover, a graph with even girth \( g \) is weakly \((g + 2)/2\)-arc transitive if and only if it is bipartite and has diameter \( g/2 \). (3) A bipartite graph with girth \( g \) is weakly \( s \)-arc transitive for all \( 0 \leq s \leq g/2 \). (4) A nonbipartite graph is weakly \( s \)-arc transitive if and only if it is \( s \)-arc transitive.

### 4 Cayley graphs of semigroups

Cayley graphs of groups have been extensively studied and many interesting results have been obtained (see[2,21,44]). The Cayley graph of a semigroup has been introduced by B.Zelinka[45].

Let \( G \) be a finite semigroup, and let \( S \) be a nonempty subset of \( G \). The Cayley graph \( \text{Cay}(G, S) \) of \( G \) relative to \( S \) is defined as the graph with vertex set \( G \) and edge set \( E(S) \) consisting of those pairs \( (x, y) \) such that \( sx = y \) for some \( s \in S \)[24].

In the investigation of the Cayley graph of semigroups it is first of all interesting to find the analogues of natural conditions which have been used in group case. All Cayley graphs of groups are transitive. A.V.Kelarev and C.E.Preager in [24] characterized transitive Cayley graphs of semigroups. A graph \( D(V, E) \) is said to be undirected if and only if, for every \( (u, v) \in E \), the edge \( (v, u) \in E \) too. It is well known that the Cayley graph \( \text{Cay}(G, S) \) of a group \( G \) is undirected if and only if \( S = S^{-1} \). A.V.Kelarev in [23] characterized undirected Cayley graphs of semigroups. From the results of A.V.Kelarev and C.E.Preager we know that the conditions for Cayley graphs of semigroups to be vertex-transitive and undirected respectively are reduced to the case of completely simple semigroups.

For semigroup terminology and notation, please refer to [22,32].
It is interesting to investigate Cayley graphs of completely simple semigroups, obtain respectively vertex-transitive and undirected Cayley graphs of these semigroups, and finally give their structure (Fan, [17]). Y.S. Zeng [46] discussed the transitivity of Cayley graphs of bands and some kind of completely simple semigroups.

**Theorem 4.1** Let \( G = I \times \Lambda \) be a rectangular band, \( S \) be a subset of \( G \). The following statements are equivalent.

1. \( \text{Cay}(G, S) \) is vertex transitive.
2. \( \text{Cay}(G, S) \) is weakly vertex transitive.
3. \( \text{Cay}(G, S) \) is undirected.
4. \( \{i | (i, \lambda) \in S\} = I \).
5. \( \text{Cay}(G, S) = \bigcup \overrightarrow{K}_{|I|} = |\lambda|\overrightarrow{K}_{|I|} \). Where \( \overrightarrow{K}_{|I|} \) is a complete graph of order \( |I| \).

See [1, 39, 40] and [25, 26, 27] for the latest results.

5 Unretractive graphs

**Open question** (Ulrich Knauer [37]): For which graphs \( X \) we have \( \text{End} X = \text{SEnd} X \)? Call a graph \( X \) is E-S unretractive if \( \text{End} X = \text{SEnd} X \); E-A unretractive if \( \text{End} X = \text{Aut} X \); S-A unretractive if \( \text{SEnd} X = \text{Aut} X \). See [28, 29] for details.

Let \( A \) be a graph, \((Y_a)_{a \in V(A)}\) be a family of graphs. The generalized lexicographic product of \( A \) and \((Y_a)_{a \in V(A)}\) is a graph, denoted by \( A[(Y_a)_{a \in V(A)}] \), with vertex set \( V = \{(a, y_a) | a \in V(A), y_a \in V(Y_a)\} \), two vertices \((a, y_a)\) and \((b, y_b)\) is adjacent if and only if \( a \) and \( b \) is adjacent in \( A \) or \( a = b \) and \( y_a \) and \( y_b \) is adjacent in \( Y_a \). The following theorem gives the structure of E-S unretractive graphs [5].

**Theorem 5.1** A graph \( X \) is E-S unretractive if and only if it is a generalized lexicographic product \( A[(Y_a)_{a \in V(A)}] \) of an E-A unretractive graph \( A \) and a family completely disconnected graphs \((Y_a)_{a \in V(A)}\).

Let \( A \) be a graph, \( a \in V(A) \). Define a graph \( X \) with \( V(A) = V(A) \cup \{a', a' \notin V(A), E(X) = E(A) \cup \{x, a' | \{x, a \} \in E(A)\} \). We say that \( X \) is a split from \( A \) on \( a \). In fact, \( X = A[(Y_a)_{a \in V(A)}] \), where \( Y_a = K_2, Y_b = K_1 \) for \( b \neq a \). Thus, in Theorem 5.1 \( X = A[(Y_a)_{a \in V(A)}] \) is equivalent to say that \( X \) is obtained from \( A \) by \( k \) splits, where \( k = \sum_{a \in V(A)} |V(Y_a) - |V(A)| \). We obtained the enumeration of E-S unretractive graphs [7].

**Corollary 5.2** Any E-S unretractive graph can be obtained from an E-A unretractive graph by \( k \) splits.
Theorem 5.3 Let $X$ be an E-A unretractive graph of order $n$. The number of the E-S unretractive graphs obtained from $A$ by $k$ splits is $\sum N(k_1, \cdots, k_m)$. Where the summation is over all partitions $k_1c_1 + \cdots + k_{m-1}c_{m-1} = k, k_m = n - (k_1 + \cdots + k_{m-1}) \geq 0$. $N(k_1, \cdots, k_m)$ is the $(k_1, \cdots, k_m)$-iterative permutation number of $C = \{c_1, \cdots, c_m\}$ on the graph $A$.

Knauer [30] defined the endomorphism spectrum and the endomorphism type of a graph, characterized trees and gave a list of graphs with given endomorphism types. S.H. Fan [15] considered bipartite graphs with diameter three and girth six and obtained their endomorphism types. The following problem was raised by Knauer in [30].

Problem 5.4 Find conditions on graphs $X$ for various unretractivities of $X$.

6 Graphs and their endomorphism monoids

In this section we first investigate the question of when graphs are determined by their endomorphism monoids up to isomorphism, then discuss the graphical strong representations of graphs, and finally give the Green’s relations for endomorphism monoids of graphs.

An early result on determination by endomorphism monoids is for partially ordered sets. It asserts that two partially ordered sets $S$ and $T$ whose endomorphism monoids are isomorphic must themselves be either isomorphic or anti-isomorphic. Analogous results hold for topological structure (Molchnov [38]). We prove that any connected bipartite graph is determined by its endomorphism monoid up to isomorphism [9].

Theorem 6.1 Let $X$ and $Y$ be two connected bipartite graphs. Then the endomorphism monoids $\text{End} X$ and $\text{End} Y$ are isomorphic if and only if graphs $X$ and $Y$ are isomorphic.

We define a graphical strong representation (GSR) of a given monoid $S$ to be a graph $X$ such that the strong endomorphism monoid $\text{SEnd} X$ of the graph $X$ is isomorphic to $S$. The question is: Which finite regular monoids have graphical strong representations? The main result is: The direct product of two regular monoids has a GSR if each factor has a GSR (Fan [13]).

Another question is: Which monoids are the endomorphism monoids of some weakly vertex transitive graphs? A result is from [16].

Theorem 6.2 If a communicative monoid $M$ is isomorphic to the endomorphism monoid of some weakly vertex transitive graph, then $M \cong (\mathbb{Z}_2)^n, n = 1$ or $n \geq 5$. 
I conjectured that the converse of this theorem is also true.

The Green’s relations of endomorphism monoids of graphs was given in (Fan [3,6]).

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**References**


