Primitive Half-transitive Graphs of Valency Twice a Prime

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Notations

All graphs here are finite, simple and regular with at least one edge. Let $\Gamma$ be a graph.

- Vertex set $V\Gamma$, edge set $E\Gamma$, arc set $A\Gamma$
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- Vertex set $V\Gamma$, edge set $E\Gamma$, arc set $A\Gamma$
- $\text{Aut}\Gamma$, automorphism group of $\Gamma$
- $G \leq \text{Aut}\Gamma$, $G$ is a subgroup of $\text{Aut}\Gamma$
Key Words

For $G \leq \text{Aut}\Gamma$, the graph $\Gamma$ is said to be $
\textit{G-vertex transitive}$, if $G$ acts transitively on $V\Gamma$;
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- $G$-primitive, if $G$ acts primitively on $V\Gamma$;
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- **\( G \)-vertex transitive**, if \( G \) acts transitively on \( V\Gamma \);
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- **\( G \)-arc transitive**, if \( G \) acts transitively on \( A\Gamma \);
- **\( G \)-primitive**, if \( G \) acts primitively on \( V\Gamma \);
- **\( G \)-half transitive**, if \( G \) acts transitively on \( V\Gamma \) and on \( E\Gamma \) but not on \( A\Gamma \);
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- half-transitive, if $\text{Aut}\Gamma$ acts transitively on $V\Gamma$ and on $E\Gamma$ but not on $A\Gamma$. 
Several Known Results

- **Tutte**: A vertex and edge transitive graph of odd valency is arc transitive (Connectivity in graphs, Univ. of Toronto Press, 1966).
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- **Holt**: A half transitive graph on 27 vertices with valency 4 (*J. Graph Theory 5* (1981)), which is the smallest one (*Alspach et al, J. Austral. Math. Soc. Ser. A 56* (1994)), and is unique up to isomorphism (*Praeger and Xu, J. Algebraic Combin. 1* (1992)).
Several Known Results

- **Holt** (1981): Question on the existence of primitive half transitive graphs (Holt, *J. Graph Theory* **5** (1981); Holton, *Discrete Math.* **38** (1982)).
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- **Praeger** and **Xu** (1993): The first ten examples of primitive half transitive graphs: one of valency 24, one of valency 48, and the others have valency 120.
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- **Du** and **Xu** (1999): The smallest primitive half transitive graph has order 165 and valency 48.
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Li, Lu and Marušič (J. Algebra 279 (2004))

- Another infinite class of primitive half transitive graphs of valencies $2(2^{2m+1} - 1)$ for $m \geq 1$. 
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**Problem:** Find all integers $k$ such that there exist primitive half transitive graphs of valency $2k$. 
Main Result

We denote by $\mathcal{V}_{ph}$ the set of integers $k$ such that there exists a primitive half transitive graph of valency $2k$. Then $2, 3, 4 \not\in \mathcal{V}_{ph}$ and $7, 12, 24, 60, 2^{2m+1} - 1 \in \mathcal{V}_{ph}$.

Theorem. If $p \geq 7$ is a prime and $p \neq 13$ then $p \in \mathcal{V}_{ph}$. 
Orbital and Graph

Let $G$ be a transitive permutation on $\Omega$. Let $\alpha, \beta \in \Omega$.

- **Paired orbitals**, $\Delta := (\alpha, \beta)^G$ and $\Delta^* = (\beta, \alpha)^G$. 
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- $(\Omega, \Delta \cup \Delta^*)$ is $G$-vertex and $G$-edge transitive;
  $(\Omega, \Delta \cup \Delta^*)$ is $G$-arc transitive $\iff \Delta(\alpha) = \Delta^*(\alpha)$
  $\iff \exists g \in N_G(G_{\alpha\beta})$ with $\beta = \alpha^g$ and $g^2 \in G_{\alpha\beta}$. 

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- $(\Omega, \Delta \cup \Delta^*)$ is $G$-vertex and $G$-edge transitive; $(\Omega, \Delta \cup \Delta^*)$ is $G$-arc transitive $\iff \Delta(\alpha) = \Delta^*(\alpha)$ $\iff \exists g \in N_G(G_{\alpha \beta})$ with $\beta = \alpha^g$ and $g^2 \in G_{\alpha \beta}$.

- $\Gamma$ is $G$-vertex and $G$-edge transitive $\Rightarrow$ $\Gamma \cong (V\Gamma, \Delta \cup \Delta^*)$ for paired orbitals $\Delta$ and $\Delta^*$. 


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- $\Delta \neq \Delta^*$ or $\Delta(\alpha) \neq \Delta^*(\alpha)$?
- $G \leq \text{Aut}(\Omega, \Delta \cup \Delta^*) =$?
Key Lemmas

Lemma A
Let $G$ be a primitive permutation group on $\Omega$, $\alpha \in \Omega$. Let $K < G_\alpha$, $K \not\trianglelefteq G_\alpha$ and $|G_\alpha : K| = l > 1$. Suppose that all subgroups of index $l$ of $G_\alpha$ are conjugate in $G_\alpha$, then

- the number of suborbits of length $l$ at $\alpha$ is equal to $|N_G(K) : K| - 1$;
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- $\Delta(\alpha) := \alpha z^{G_\alpha}$ is a suborbit with length $l$ at $\alpha$, where $z \in N_G(K) \setminus K$; $\Delta(\alpha)$ is self-paired iff $z^2 \in K$.
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- the number of suborbits of length $l$ at $\alpha$ is equal to $|N_G(K) : K| - 1$;
- $\Delta(\alpha) := \alpha^z G_\alpha$ is a suborbit with length $l$ at $\alpha$, where $z \in N_G(K) \setminus K$; $\Delta(\alpha)$ is self-paired iff $z^2 \in K$;
- $(\Omega, \Delta \cup \Delta^*)$ is $G$-arc transitive iff $z^2 \in K$; there exist primitive $G$-half transitive graphs of valency $2l$ iff $N_G(K)/K$ is not an elementary abelian 2-group.
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**Lemma B**
Let $p \geq 7$ be a prime, and let $x \in S_p$ be of order $p$. Then

- $x$ is a $p$-cycle;
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- $N_{S_p}(\langle y \rangle) = N_{S_{p-1}}(\langle y \rangle)$;
- $N_{S_{p-1}}(\langle y \rangle)/\langle y \rangle$ is an elementary abelian 2-group iff $p = 7$ or 13.
Construction

Let $p \geq 7$ be a prime.

- Take a $p$-cycle $x \in S_p$ and a $(p - 1)$-cycle $y \in S_{p-1}$ such that $y$ normalizes $\langle x \rangle$. 
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- Take a $p$-cycle $x \in S_p$ and a $(p - 1)$-cycle $y \in S_{p-1}$ such that $y$ normalizes $\langle x \rangle$.
- \( H = \langle x, y \rangle \), \( K = \langle y \rangle \) and \( \Omega := \{ Hg \mid g \in S_p \} \).
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- $\Delta = (\alpha, \alpha^z)^{S_p}$, $\Delta^* = (\alpha, \alpha^{-1}z)^{S_p}$, $\alpha := H \in \Omega$, $z \in N_{S_{p-1}}(K) \setminus K$. Then $\Delta(\alpha) = \alpha^zH$ and $\Delta^*(\alpha) = \alpha^{-1}zH$ are paired suborbits of length $p$. 
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- $\Gamma(p, z) := (\Omega, \Delta \cup \Delta^*)$. 
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- \( \Gamma(7, z) \) and \( \Gamma(13, z) \) are arc transitive.
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- **Γ(7, z)** and **Γ(13, z)** are arc transitive.
- **Γ(\(p, z\))** is \(S_p\)-half transitive of valency \(2p\) if \(p \geq 11\), \(p \neq 13\) and \(z^2 \notin K\) (there always exists such a \(z\)).
$\Gamma(p, z)$

- $\Gamma(p, z)$ is an $S_p$-primitive Cayley graph of $S_{p-2}$.
- $\Gamma(p, z)$ is $S_p$-arc transitive of valency $p$ if $z^2 \in K$.
- $\Gamma(7, z)$ and $\Gamma(13, z)$ are arc transitive.
- $\Gamma(p, z)$ is $S_p$-half transitive of valency $2p$ if $p \geq 11$, $p \neq 13$ and $z^2 \not\in K$ (there always exists such a $z$).
- $\text{Aut}\Gamma(p, z) = S_p$. 

- p. 13/16
Conclusion

For each prime $p \geq 7$ with $p \neq 13$, there exists at least one primitive half transitive graph of valency $2p$. 
Problems

- Noting that the automorphism of each graph given by our construction has a solvable vertex stabilizer. Then an interesting problem arises: Is there a primitive half-transitive graph of valency $2p$ such that its automorphism group has insolvable vertex stabilizers?

- We know affirmatively that 2, 3 and 4 are not members of $\mathcal{V}_{ph}$. How about 5, 13 and other small positive integers?
Thank You!
References


