Unimodality Problems in Combinatorics

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"Combinatorialists love to prove that counting sequences are unimodal." — D.Zeilberger

Outline

- Unimodal and log-concave sequences
- Pólya frequency sequences
- Polynomials with only real zeros
- Linear transformations preserving log-concavity
- Log-convex sequences
- Conjectures and open problems

Unimodal and log-concave sequences

Def. Let $\{a_0, a_1, \ldots\}$ be a sequence of positive numbers.

It is *unimodal* (**UM**) if $a_0 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots$. ($m$ is called a *mode* of the sequence.)

It is *log-concave* (**LC**) if $a_{i-1}a_{i+1} \leq a_i^2$ for all $i > 0$.

- LC $\iff$ a_i/a_{i-1} is decreasing $\Longrightarrow$ UM.

Ex. $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ has a mode $n/2$ or two modes $(n \pm 1)/2$.

Proof. $\binom{n}{i}/\binom{n}{i-1} = (n-i+1)/i$. $\square$

Prop. Suppose that $\sum_{i=0}^{n} a_i x^i = a_n \prod_{j=1}^{n}(x + r_j) \in \mathrm{RZ}$. Then

(1) (Newton) $a_i^2 \geq a_{i-1}a_{i+1}\dfrac{(i+1)(n-i+1)}{i(n-i)}$.

(2) (Darroch) $|\text{mode } (a_i) - M| < 1$.

(3) (Benoumhani) $\lfloor M \rfloor \leq \text{mode } (a_i) \leq \lceil M \rceil$.

Therefore $a_0, \ldots, a_n$ is LC and UM with at most two modes.

- Let **RZ** denote the set of polynomials with only **R**eal **Z**eros.
- $M = \dfrac{\sum_{i=1}^{n} i a_i}{\sum_{i=0}^{n} a_i} = \sum_{j=1}^{n} \dfrac{1}{r_j + 1}$.

Ex. $\sum_{i=0}^{n}\binom{n}{i}x^i = (x+1)^n$. $M = \frac{n}{2}$.

Ex. The (signless) Stirling number of the first kind $c(n,k)$ is the number of permutations of $[n] := \{1,2,\ldots,n\}$ with exactly k cycles.

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1).$$

$$M = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \sim \ln n + \gamma.$$

Erdös showed that the sequence $c(n,0)$, $c(n,1)$, . . . , $c(n,n)$ has a unique mode for $n \geq 3$.

Ex. The Stirling number of the second kind $S(n,k)$ is the number of partitions of $[n]$ into k blocks. Clearly, $S(n,k) = kS(n-1,k) + S(n-1,k-1)$.

Let $S_n(x) = \sum_{k=0}^{n} S(n,k)x^k$. Then $S_n(x) = xS_{n-1}(x) + xS'_{n-1}(x)$.

Harper showed that $S_n(x) \in \mathrm{RZ}$. Denote

$$\tau_n = \frac{\sum_{k=1}^{n} kS(n,k)}{\sum_{k=1}^{n} S(n,k)} = \frac{B_{n+1}}{B_n} - 1, \quad B_n = \sum_{k=1}^{n} S(n,k).$$

Canfield showed that $S(n,0), S(n,1), \ldots, S(n,n)$ is unimodal with at most two modes K_n and $K_n + 1$ where $K_n \sim \frac{n}{\ln n}$. It is unknown how to determine K_n as a function of n and whenever $S(n,0), S(n,1), \ldots, S(n,n)$ has the unique mode.

Engel conjectured that τ_n is concave, i.e., $\tau_{n-1} + \tau_{n+1} \leq 2\tau_n$.

Griggs verified the conjecture for $2 \leq n \leq 1200$.

Canfield proved the conjecture for sufficiently large n.

Totally positive matrices and Pólya frequency sequencess

Def. An infinite matrix is **TP** if its minors are all nonnegative. Given an infinite sequence $\{a_0, a_1, a_2, \ldots\}$, define its Toeplitz matrix

$$A = (a_{j-i})_{i,j\geq 0} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdot \\ & a_0 & a_1 & a_2 & \cdot \\ \mathrm{O} & & a_0 & a_1 & \cdot \\ & & & a_0 & \cdot \\ & & & & \cdot \end{pmatrix}.$$

A finite sequence $\{a_0, a_1, \ldots, a_n\} \cong \{a_0, a_1, \ldots, a_n, 0, 0, \ldots\}$.

Def. The sequence $\{a_i\}$ is **PF** if the matrix A is TP.

- PF$\Longrightarrow$LC$\Longrightarrow$UM.

Aissen-Schoenberg-Whitney Thm. Let $a_i \geq 0$. Then

$$a_0, \ldots, a_n \text{ is PF} \iff \sum_{i=0}^{n} a_i x^i \in \mathrm{RZ}.$$

- Denote $\sum_{i=0}^{n} a_i x^i \in \mathrm{PF}$ if $a_0, \ldots, a_n$ is PF.

Schoenberg-Edrei Thm. Let $a_i \geq 0$. Then

$$a_0 = 1, a_1, a_2, \ldots \text{ is PF} \iff \sum_{i \geq 0} a_i x^i = \frac{\prod_{j \geq 1}(1 + \alpha_j x)}{\prod_{j \geq 1}(1 - \beta_j x)}\, e^{\gamma x},$$

where $\gamma, \alpha_j, \beta_j \geq 0$ and $\sum(\alpha_j + \beta_j) < \infty$.

Polynomials with only real zeros

Suppose that $f(x) = \alpha \prod_{i=1}^{n}(x - r_i), \quad r_n \le \cdots \le r_1$

and $g(x) = \beta \prod_{j=1}^{m}(x - s_j), \quad s_m \le \cdots \le s_1.$

We say that g **interlaces** f if $m = n - 1$ and

$$r_n \le s_{n-1} \le r_{n-1} \le \cdots \le r_2 \le s_1 \le r_1.$$

We say that g **alternates** f if $m = n$ and

$$s_n \le r_n \le s_{n-1} \le \cdots \le r_2 \le s_1 \le r_1.$$

Ex. $f \in \mathrm{RZ} \Longrightarrow f' \preccurlyeq_{\mathrm{int}} f; \quad f \in \mathrm{PF} \Longrightarrow f \preccurlyeq_{\mathrm{alt}} xf'.$

Thm. (Wang & Yeh, J. Combin. Theory Ser. A, 2005)

Let $F(x) = (bx + a)f(x) + (dx + c)g(x)$. Suppose that

(1) f and g have leading coefficients of the same sign.

(2) $f, g \in \mathrm{RZ}$.

(3) $g \preccurlyeq f$.

If $ad \geq bc$, then $F(x) \in \mathrm{RZ}$.

Coro. Let $F(x) = (ax + b)f(x) + x(cx + d)g(x)$. Suppose that $f, g \in \mathrm{PF}$ and $g \preccurlyeq_{\mathrm{int}} f$. If $ad \geq bc$, then $F \in \mathrm{RZ}$.

Thm. (Liu & Wang, Adv. in Appl. Math., 2006) Suppose that

(1) $F(x) = u(x)f(x) + v(x)g(x)$ and $\deg F = \deg f$ or $\deg f + 1$.

(2) $f, g \in \mathrm{RZ}$ and $g \preccurlyeq f$.

(3) F and g have leading coefficients of the same sign.

If $v(r) \le 0$ whenever $f(r) = 0$, then $F \in \mathrm{RZ}$ and $f \preccurlyeq F$.

Ex. Orthogonal polynomials: $p_n(x) = (a_n x + b_n)p_{n-1}(x) - c_n p_{n-2}(x)$.

Stirling polynomials: $S_n(x) = xS_{n-1}(x) + xS'_{n-1}(x)$.

Eulerian polynomials: $A_n(x) = nxA_{n-1}(x) + x(1-x)A'_{n-1}(x)$.

Narayana polynomials: $N_n(x) = \sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1}x^k$.

$$(n+1)N_n(x) = (2n-1)(1+x)N_{n-1}(x) - (n-2)(1-x)^2 N_{n-2}(x).$$

Thm. (Ma & Wang, 2006) Suppose that f and F have leading coefficients of the same sign and $F(x) = u(x)f(x) + v(x)f'(x)$. If $f \in \mathrm{RZ}$ and $v(r) \leq 0$ whenever $f(r) = 0$, then $F \in \mathrm{RZ}$ and $f \preccurlyeq F$. Let r be a zero of f with the multiplicity m. Then the multiplicity of r as a zero of F is

(1) $m - 1$ if $v(r) \neq 0$; or

(2) m if $v(r) = 0$ but $u(r) + mv'(r) \neq 0$; or

(3) $m + 1$ if $v(r) = 0$ and $u(r) + mv'(r) = 0$.

In particular, if all zeros of f are simple and $v(r) \neq 0$ or $u(r) + v'(r) \neq 0$ whenever $f(r) = 0$, then all zeros of F are simple.

Ex. $G_{n+2}(x) = x(nx + 2)G_{n+1}(x) + x\left(1 - x^2\right)G'_{n+1}(x), \quad G_1(x) = 1.$

Knuth: $G_n(x) = \left(\frac{1+x}{2}\right)^{n-1}(1 + w)^{n+1}A_n\left(\frac{1-w}{1+w}\right), \quad w = \sqrt{\frac{1-x}{1+x}}.$

Wilf: $G_n(x) \in \mathrm{RZ}[-1, 0]$. **Bóna:** The multiplicity of $x = -1$ is $\left\lfloor \frac{n}{2} \right\rfloor - 1$.

Linear transformations preserving log-concavity

Def. Let $a(n,k) \geq 0$. We say that the linear transformation

$$z_n = \sum_{k=0}^{n} a(n,k) x_k, \quad n = 0, 1, 2, \ldots.$$

is **PLC** if it preserves the log-concavity. We say that

$$z_n = \sum_{k=0}^{n} a(n,k) x_k y_{n-k}, \quad n = 0, 1, 2, \ldots.$$

is **double PLC** if $\{x_k\}$ and $\{y_k\}$ are LC implies $\{z_n\}$ is LC.

- double PLC $\Longrightarrow$ PLC.

Def. Let $\{a(n,k)\}_{0\le k\le n}$ be a triangle of nonnegative numbers. For $n \ge r$, define the polynomial

$$A_n(r;q) = \sum_{k=r}^{n} a(n,k)q^k.$$

We say that the triangle $\{a(n,k)\}$ is **LC-positive** if for each $r \ge 0$, the sequence of polynomials $\{A_n(r;q)\}$ is q-LC in n, i.e.,

$$A_n^2(r;q) - A_{n-1}(r;q)A_{n+1}(r;q)$$

has nonnegative coefficients as a polynomial in q.

Ex. $a(n,k) \equiv 1$.

Ex. $a(n,k) = \binom{n}{k}$.

Thm. (Wang & Yeh, J. Combin. Theory Ser. A, 2006)

(1) $\{a(n,k)\}$ is LC-positive $\Longrightarrow \{a(n,k)\}$ is PLC.

(2) $\{a(n,k)\}$ and $\{a^*(n,k)\}$ are LC-positive $\Longrightarrow \{a(n,k)\}$ is double PLC.

- $a^*(n,k) := a(n,n-k)$.

Ex. (1) $z_n = \sum_{k=0}^{n} x_k y_{n-k}$.

(2) $z_n = \sum_{k=0}^{n} \binom{n}{k} x_k y_{n-k}$. (Walkup)

(3) $z_n = \sum_{k=0}^{n} \binom{a+n}{b+k} x_k y_{n-k}$. (Wang)

(4) $z_n = \sum_{k=0}^{n} \binom{a-n}{b-k} x_k y_{n-k}$. (Wang)

(5) $z_n = \sum_{k=0}^{n} \binom{n}{k}\binom{a-n}{b-k} x_k y_{n-k}$. (Pemantle, Liggett)

Log-convex sequences

Def. A sequence $\{z_n\}$ is *log-convex* (**LCX**) if $z_{n-1}z_{n+1} \geq z_n^2$.

Ex. Central binomial coefficients $b(n) = \binom{2n}{n}$.

Ex. Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Prop. If both $\{x_n\}$ and $\{y_n\}$ are LCX, then so is $\{x_n + y_n\}$.

Davenport and Pólya Thm. If both $\{x_n\}$ and $\{y_n\}$ are LCX, then so is the sequence $\{z_n\}$ defined by $z_n = \sum_{k=0}^{n} \binom{n}{k} x_k y_{n-k}$.

Ex. Euler numbers: $2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}$.

Ex. Bell numbers: $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$.

Thm. (Liu & Wang, 2006) Suppose that $\{z_n\}_{n\geq 0}$ satisfies

(1) $a_n z_{n+1} = b_n z_n + c_n z_{n-1}$, where $a_n, b_n, c_n > 0$.

(2) z_0, z_1, z_2, z_3 is log-convex.

(3) $a_n \lambda_{n-1}\lambda_{n+1} - b_n \lambda_{n-1} - c_n \geq 0$ for $n \geq 2$.

Then $\{z_n\}$ is LCX.

- $\lambda_n = \frac{b_n + \sqrt{b_n^2 + 4a_n c_n}}{2a_n}$ is the unique positive root of $a_n \lambda^2 = b_n \lambda + c_n$.

Ex. Fine numbers: $2(n+1)f_n = (7n-5)f_{n-1} + 2(2n-1)f_{n-2}$.

Ex. Motzkin numbers: $(n+3)M_{n+1} = (2n+3)M_n + 3nM_{n-1}$.

Thm. (Liu & Wang, 2006) Suppose that z_0, z_1, z_2 is LCX and

$$(\alpha_1 n + \alpha_0) z_{n+1} = (\beta_1 n + \beta_0) z_n - (\gamma_1 n + \gamma_0) z_{n-1}.$$

Denote $A = \begin{vmatrix} \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{vmatrix}, \quad B = \begin{vmatrix} \gamma_0 & \gamma_1 \\ \alpha_0 & \alpha_1 \end{vmatrix}, \quad C = \begin{vmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \end{vmatrix}.$

Then $\{z_n\}_{n \geq 0}$ is LCX if one of the following conditions holds.

(1) $B, C \geq 0$.

(2) $B < 0, C > 0, AC \geq B^2$ and $z_0 B + z_1 C \geq 0$.

(3) $B > 0, C < 0, AC \leq B^2$ and $z_0 B + z_1 C \geq 0$.

Ex. Central Delannoy numbers: $n z_n = 3(2n-1) z_{n-1} - (n-1) z_{n-2}$.

Ex. Schröder numbers: $(n+2) z_{n+1} = 3(2n+1) z_n - (n-1) z_{n-1}$.

Coro. Suppose that $\{z_n\}_{n\geq 0}$ satisfies $az_{n+1} = bz_n - cz_{n-1}$, where $a, b, c > 0$. If z_0, z_1, z_2 is LCX (resp. LC), then so is $\{z_n\}_{n\geq 0}$.

Coro. Suppose that $\{z_n\}_{n\geq 0}$ satisfies $az_{n+1} = bz_n + cz_{n-1}$, where $a, b, c > 0$. If z_0, z_1, z_2 is LCX (resp. LC), then $\{z_{2n}\}$ is LCX (resp. LC) and $\{z_{2n+1}\}$ is LC (resp. LCX).

Proof. We have $a^2 z_{n+2} = (b^2 + 2ac)z_n - c^2 z_{n-2}$ for $n \geq 2$. □

Ex. Fibonacci sequence $\{F_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$ satisfies

$$F_{n+1} = F_n + F_{n-1}.$$

$\{F_{2n}\} = \{1, 2, 5, 13, \ldots\}$ is LCX.

$\{F_{2n-1}\} = \{1, 3, 8, 21, \ldots\}$ is LC.

Conjectures and open problems

- (Almkvist) The polynomial $\prod_{i=1}^{n} \dfrac{1-q^{ri}}{1-q^{i}}$ is unimodal if

(1) r is even and $n \geq 1$; or

(2) r is odd and $n \geq 11$.

The conjecture holds for $r = 2, \ldots, 20, 100, 101$.

Ex. $(1+q)(1+q^2)\cdots(1+q^n)$, no combinatorial proof.

- (Read) chromatic polynomials of finite graphs
- (Rota) rank numbers of finite geometric lattices

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Thank you for your attention!