Stable Equivalence over Symmetric Functions

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Part I

Background

- Stable equivalence
- Schur function
- Kuperberg’s question

Stable equivalence

R: a commutative ring with unit

M: an $n \times k$ matrix over R

M^T: the transpose of M

[Kuperberg, 2002] introduced the concept of stable equivalence.

M' is called a stably equivalent form of M if M' can be obtained from M under the following operations:

Stable equivalence

general row operations

$$M \rightarrowtail AM,\ A \text{ is an } n \times n \text{ invertible matrix.}$$

general column operations

$$M \rightarrowtail MB,\ B \text{ is a } k \times k \text{ invertible matrix.}$$

stabilization and its inverse

$$M \rightarrowtail \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & M \end{array}\right), \quad \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & M \end{array}\right) \rightarrowtail M.$$

Partitions

An integer partition is a decreasing sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$, and we say λ is a partition of n if $\sum_i \lambda_i = n$, denoted $\lambda \vdash n$.

The number of positive parts of λ is called the length of λ, denoted $\ell(\lambda)$.

Example: $\lambda = (3, 1, 1)$ is a partition of 5 and $\ell(\lambda) = 3$.

Young Diagram

A Young diagram of λ can be defined as an array of squares justified from the top and left corner with $\ell(\lambda)$ rows and λ_i squares in row i.

A square (i, j) in the diagram is the square in row i from the top and column j from the left. The content of (i, j), denoted $\tau(i, j)$, is given by $j - i$.

Skew partitions

Given two partitions λ and μ, we say $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i.

If $\mu \subseteq \lambda$, we define a skew partition λ/μ, whose Young diagram (skew Young diagram) is obtained from the Young diagram of λ by peeling off the Young diagram of μ from the upper left corner.

Note: When $\mu = \emptyset$, λ/μ becomes the ordinary partition λ.

Semistandard Young Tableaux

A semistandard Young tableau (SSYT) of shape λ is a filling of the Young diagram such that the elements are strictly increasing in each column and weakly increasing in each row.

1	1	1	3	4	4
2	4	4	5	5	
5	5	7			
6	9	9			

Figure 1: SSYT of shape 6533

Semistandard Young Tableaux

Given an SSYT T, for each cell (i, j) of T in row i and column j, denote the element in this cell by $T(i, j)$, and define

$$x^T = \prod_{(i,j)\in T} x_{T(i,j)}$$

For the example in figure 2, we have

$$x^T = x_1^3 x_2 x_3 x_4^4 x_5^4 x_6 x_7 x_9^2$$

Semistandard Young Tableaux

A semistandard Young tableau (SSYT) of skew shape λ/μ is a filling of the Young diagram λ/μ such that the elements are strictly increasing in each column and weakly increasing in each row. The definition of x^T is the same as that of the ordinary shape.

			3	4	4
	1	4	7	7	
2	2	6			
3	8	8			

Figure 2: SSYT of shape $6533/31$

Schur Function

The Schur function $s_{\lambda/\mu}(X)$ is defined as follows:

$$s_{\lambda/\mu}(X) = \sum_{T} x^T$$

summed over all SSYTs T of shape λ/μ.

$$\begin{aligned} s_{21}(x_1, x_2, x_3) &= x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 \\ &\quad +x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3 \end{aligned}$$

Schur Function

When $\lambda = n$ and $\mu = \emptyset$, $s_{\lambda/\mu}(X)$ is the n-th complete symmetric function, denoted by h_n, where

$$h_n = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n} \quad n \geq 1.$$

When $\lambda = 1^n$ and $\mu = \emptyset$, $s_{\lambda/\mu}(X)$ is the n-th elementary symmetric function, denoted by e_n,

where $e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n} \quad n \geq 1.$

Jacobi-Trudi identity

- Jacobi-Trudi identity:

$$s_{\lambda/\mu} = \det(J_{\lambda/\mu}) = \det(h_{\lambda_i-\mu_j-i+j})$$

where $h_0 = 1$ and $h_k = 0$ for $k < 0$

- Dual Jacobi-Trudi identity:

$$s_{\lambda/\mu} = \det(D_{\lambda/\mu}) = \det(e_{\lambda'_i-\mu'_j-i+j})$$

where $e_0 = 1$ and $e_k = 0$ for $k < 0$

Kuperberg's Question

Are $J_{\lambda/\mu}$ and $D_{\lambda/\mu}$ stably equivalent over the ring of symmetric functions?

Kuperberg's Theorem [2002]:

$J_{\lambda/\mu}$ and $D_{\lambda/\mu}$ are stably equivalent over the ring of polynomials.

Part II

Main result

- Main theorem
- Cutting strips
- General result

Main theorem

$J_{\lambda/\mu}$ and $D^T_{\lambda/\mu}$ are stably equivalent over the ring of symmetric functions.

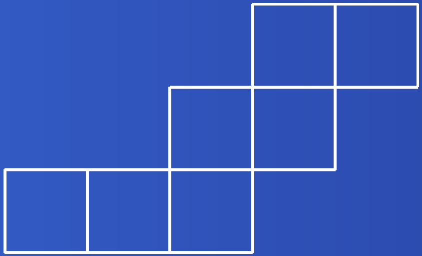
Remark:

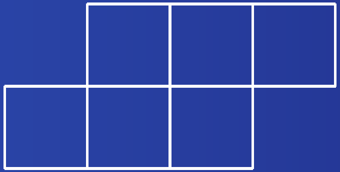
In fact, Kuperberg proved that $J_{\lambda/\mu}$ and $D^T_{\lambda/\mu}$ (not $D_{\lambda/\mu}$) are stably equivalent over the ring of polynomials.

Border Strip

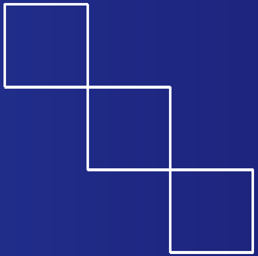
A border strip is a diagram of skew shape with an edgewise connected set of boxes that contains no 2×2 block of boxes.

Example: The first diagram is a border strip; while the last two are not.





2×2 squares



not connected

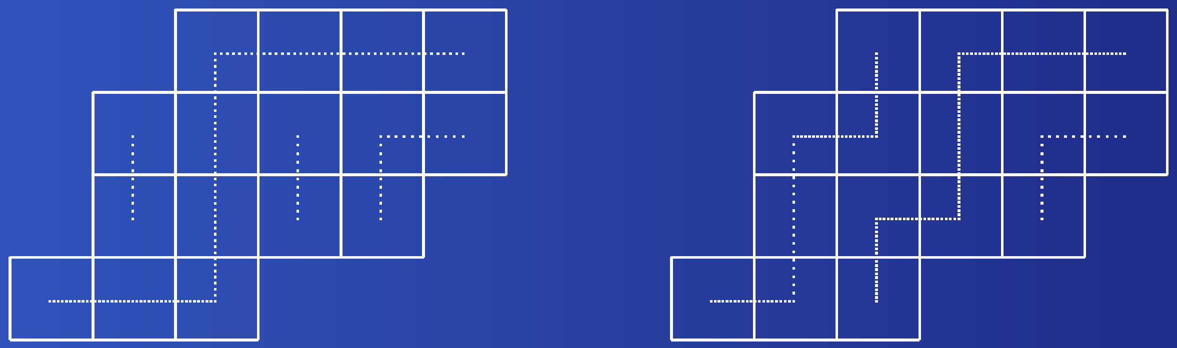
Outside decomposition

A border strip decomposition of λ/μ is a partitioning of the boxes of λ/μ into pairwise disjoint border strips.

A border strip decomposition of λ/μ is said to be an outside decomposition:

If every strip in the decomposition has an initial square on the left or bottom perimeter of the diagram and a terminal square on the right or top perimeter.

Outside decomposition



border strip decomposition outside decomposition

Cutting strip

The cutting strip ϕ (introduced by Chen, Yan and Yang) of an outside decomposition **D** of λ/μ is defined as follows:

It is a border strip of length k, for $i = 1, 2, \ldots, k-1$ the i-th square in ϕ keeps the same direction as the i-th diagonal of λ/μ with respect to **D**.

Cutting strip

Given a border strip θ of **D**, let $p(\theta)$ denote the lower left-hand square of θ, and let $q(\theta)$ denote the upper right-hand square of θ.

Suppose that λ/μ is an edgewise connected skew partition with d non-empty diagonals. Then there is a one-to-one correspondence between the outside decompositions of λ/μ and the set of border strips with d boxes.

Cutting strip

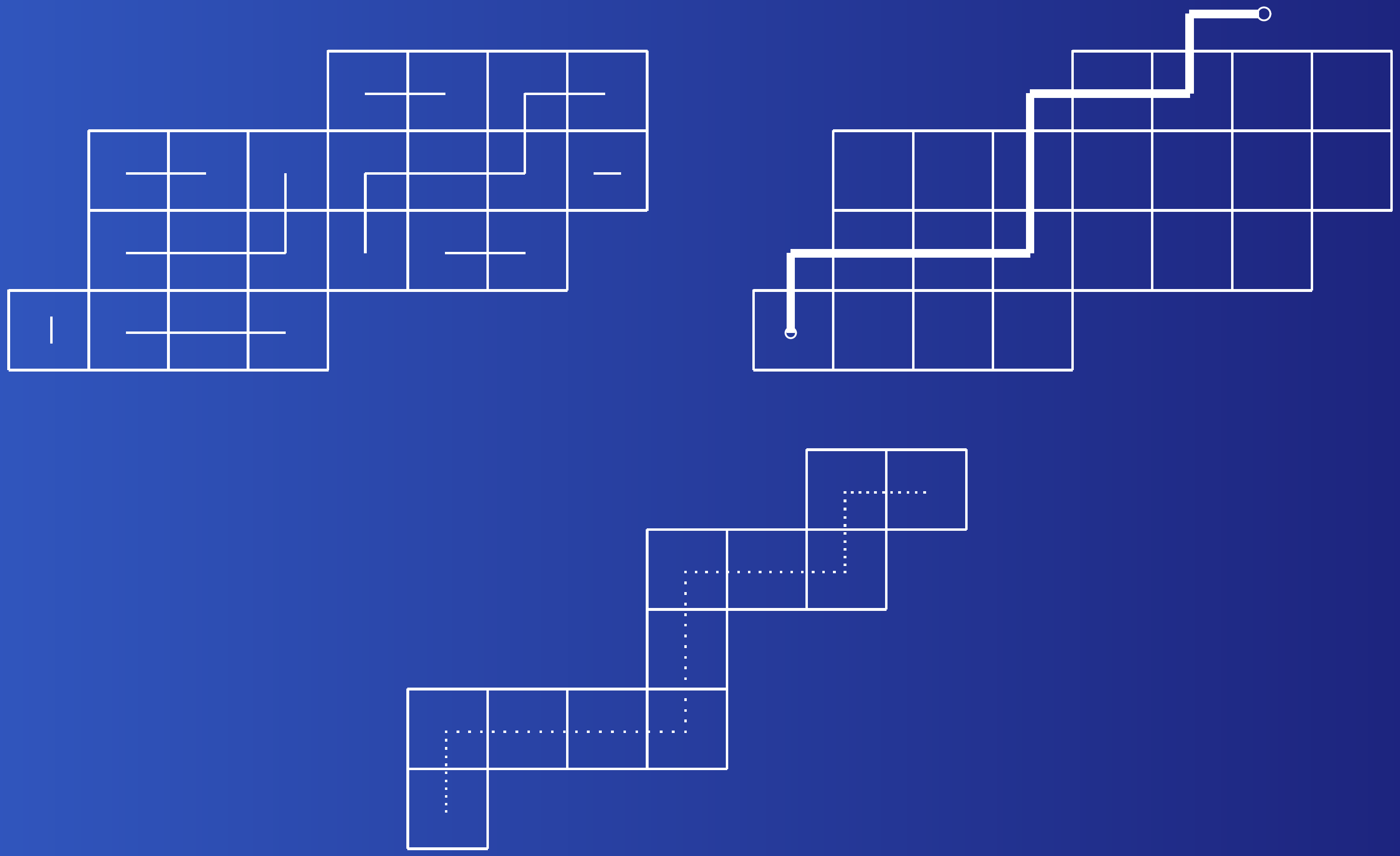


Figure 3: The cutting strip of $(8, 8, 7, 4)/(4, 1, 1)$

Giambelli type determinant formula

Theorem *(Hamel and Goulden)*
*For an outside decomposition **D** with k border strips $\theta_1, \theta_2, \ldots, \theta_k$, we have*

$$s_{\lambda/\mu} = det(M_D) = det(s_{[\tau(p(\theta_i)),\tau(q(\theta_j))]})_{i,j=1}^{k} \qquad (**)$$

A strip $[\alpha, \beta]$ is defined by the following rule:

- *$[\alpha, \beta]$ is the segment of ϕ from the square with content α to the square with content β if $\alpha \leq \beta$*
- $[\alpha + 1, \alpha] = \emptyset$
- *$[\alpha, \beta]$ is undefined if $\alpha > \beta + 1$*

General result

Let D and D' be two outside decompositions of the edgewise connected skew diagram λ/μ. Then the Giambelli-type matrices M_D and $M_{D'}$ are stably equivalent over the ring of symmetric functions.

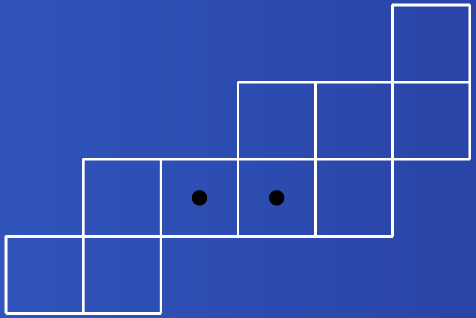
$J_{\lambda/\mu}$: horizontal decomposition

$D_{\lambda/\mu}$: vertical decomposition

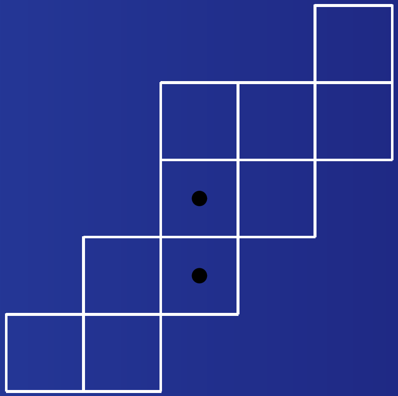
Key Lemma

Let I and J be two skew diagrams. Then

$$s_I s_J = s_{I \blacktriangleright J} + s_{I \uparrow J}.$$



$I \blacktriangleright J$

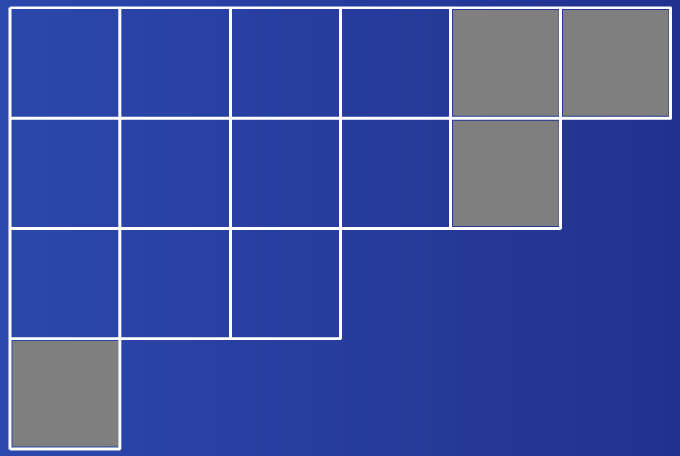


$I \uparrow J$

Part III

One example

Take $\lambda/\mu = (6,5,3,1)/(4,4,3)$.

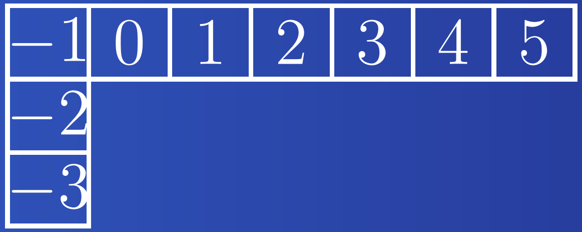


<table>
<tr><th>Cutting strip and outside decomposition</th><th>Canonical form of Giambelli-type matrix</th></tr>
<tr><td>$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \end{array}$
$\{[4,5],[3,3],[1,0],[-3,-3]\}$</td><td>$\begin{pmatrix} s_2 & 1 & \cdot & \cdot \\ s_3 & s_1 & \cdot & \cdot \\ s_5 & s_3 & 1 & \cdot \\ s_9 & s_7 & s_4 & s_1 \end{pmatrix}$</td></tr>
</table>

Cutting strip and outside decomposition	Canonical form of Giambelli-type matrix
$\begin{array}{\|c\|c\|c\|c\|c\|c\|c\|c\|}\hline -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline -3 \\ \cline{1-1} \end{array}$ $\{[4,5],[3,3],[1,0],[-3,-3]\}$	$\begin{pmatrix} s_2 & 1 & \cdot & \cdot \\ s_3 & s_1 & \cdot & \cdot \\ s_5 & s_3 & 1 & \cdot \\ s_{81} & s_{61} & s_{31} & s_1 \end{pmatrix}$

$$s_1 s_k = s_{k+1} + s_{k1}$$

Cutting strip and outside decomposition	Canonical form of Giambelli-type matr

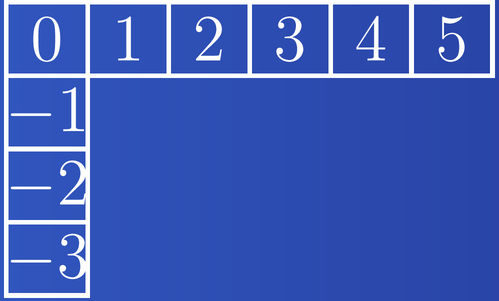


$$\{[4,5],[3,3],[1,0],[-1,-2],[-3,-3]\}$$

$$\left(\begin{array}{cccc} s_2 & 1 & \cdot & \cdot \\ s_3 & s_1 & \cdot & \cdot \\ s_5 & s_3 & 1 & \cdot \\ s_7 & s_5 & s_2 & 1 \\ s_{71^2} & s_{51^2} & s_{21^2} & s_{1^2} \end{array}\right.$$

$$s_k s_{1^2} = s_{k+1,1} + s_{k,1^2}$$

Cutting strip and outside decomposition



$$\begin{array}{|c|c|c|c|c|c|}\hline 0&1&2&3&4&5\\ \hline -1&&&&&\\ \cline{1-1} -2&&&&&\\ \cline{1-1} -3&&&&&\\ \cline{1-1}\end{array}$$

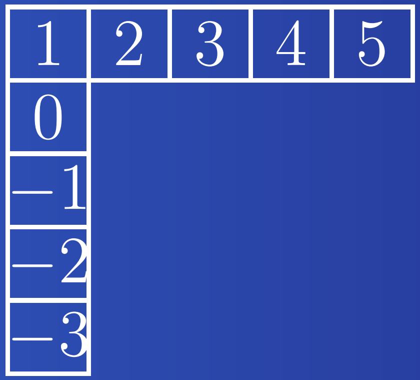
$$\{[4,5],[3,3],[1,0],[0,-1],$$
$$[-1,-2],[-3,-3]\}$$

Canonical form of Giambelli-type matrix

$$\begin{pmatrix} s_2 & 1 & \cdot & \cdot & \cdot & \cdot \\ s_3 & s_1 & \cdot & \cdot & \cdot & \cdot \\ s_5 & s_3 & 1 & \cdot & \cdot & \cdot \\ s_6 & s_4 & s_1 & 1 & \cdot & \cdot \\ s_{61} & s_{41} & s_{1^2} & s_1 & 1 & \cdot \\ s_{61^3} & s_{41^3} & s_{1^4} & s_{1^3} & s_{1^2} & s_1 \end{pmatrix}$$

$$s_k s_1 = s_{k,1} + s_{k+1}, \qquad s_k s_{1^3} = s_{k,1^3} + s_{k+1,1^2}$$

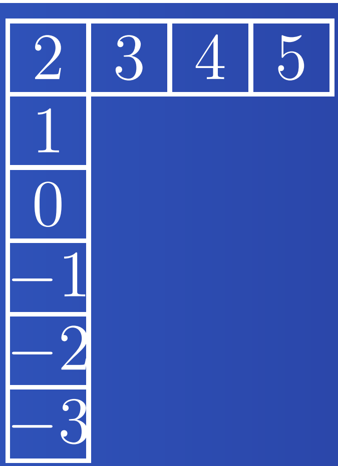
Cutting strip and outside decomposition

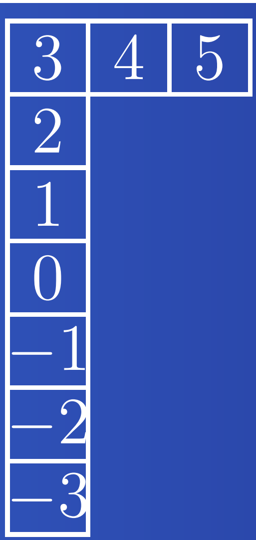


$$\{[4,5],[3,3],[0,-1],[-1,-2],[-3,-3]\}$$

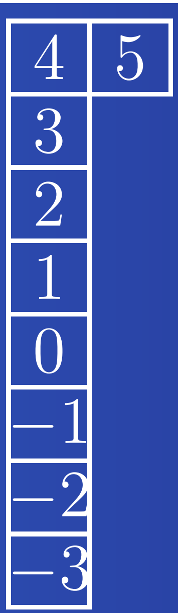
Canonical form Giambelli-type ma

$$\left(\begin{matrix} s_2 & 1 & \cdot & \cdot \\ s_3 & s_1 & \cdot & \cdot \\ s_{51} & s_{31} & 1 & \cdot \\ s_{51^2} & s_{31^2} & s_1 & 1 \\ s_{51^4} & s_{31^4} & s_{1^3} & s_{1^2} \end{matrix}\right.$$

Cutting strip and outside decomposition	Canonical form of Giambelli-type matrix
 $\{[4,5],[3,3],[2,1],$ $[0,-1],[-1,-2],[-3,-3]\}$	$\begin{pmatrix} s_2 & 1 & \cdot & \cdot & \cdot & \cdot \\ s_3 & s_1 & \cdot & \cdot & \cdot & \cdot \\ s_4 & s_2 & 1 & \cdot & \cdot & \cdot \\ s_{41^2} & s_{21^2} & s_{1^2} & 1 & \cdot & \cdot \\ s_{41^3} & s_{21^3} & s_{1^3} & s_1 & 1 & \cdot \\ s_{41^5} & s_{21^5} & s_{1^5} & s_{1^3} & s_{1^2} & s_1 \end{pmatrix}$

Cutting strip and outside decomposition	Canonical form of Giambelli-type matrix
 $\{[4,5],[3,3],[2,1],$ $[0,-1],[-1,-2],[-3,-3]\}$	$\begin{pmatrix} s_2 & 1 & \cdot & \cdot & \cdot & \cdot \\ s_3 & s_1 & \cdot & \cdot & \cdot & \cdot \\ s_{31} & s_{1^2} & 1 & \cdot & \cdot & \cdot \\ s_{31^3} & s_{1^4} & s_{1^2} & 1 & \cdot & \cdot \\ s_{31^4} & s_{1^5} & s_{1^3} & s_1 & 1 & \cdot \\ s_{31^6} & s_{1^7} & s_{1^5} & s_{1^3} & s_{1^2} & s_1 \end{pmatrix}$

Cutting strip and outside decomposition



$$\{[3,5],[2,1],[0,-1],[-1,-2],[-3,-3]\}$$

Canonical form o Giambelli-type ma

$$\left(\begin{matrix} s_{21} & \cdot & \cdot & \cdot \\ s_{21^2} & 1 & \cdot & \cdot \\ s_{21^4} & s_{1^2} & 1 & \cdot \\ s_{21^5} & s_{1^3} & s_1 & 1 \\ s_{21^7} & s_{1^5} & s_{1^3} & s_{1^2} \end{matrix}\right.$$

Cutting strip and outside decomposition

$$\begin{array}{|c|}\hline 5\\ \hline 4\\ \hline 3\\ \hline 2\\ \hline 1\\ \hline 0\\ \hline -1\\ \hline -2\\ \hline -3\\ \hline \end{array}$$

$$\{[5,5],[3,4],[2,1],[0,-1],[-1,-2],[-3,-3]\}$$

Canonical form of Giambelli-type matrix

$$\begin{pmatrix} s_1 & 1 & \cdot & \cdot & \cdot & \cdot \\ s_{1^3} & s_{1^2} & \cdot & \cdot & \cdot & \cdot \\ s_{1^4} & s_{1^3} & 1 & \cdot & \cdot & \cdot \\ s_{1^6} & s_{1^5} & s_{1^2} & 1 & \cdot & \cdot \\ s_{1^7} & s_{1^6} & s_{1^3} & s_1 & 1 & \cdot \\ s_{1^9} & s_{1^8} & s_{1^5} & s_{1^3} & s_{1^2} & s_1 \end{pmatrix}$$

THANK YOU