# Erdös-Ko-Rado Type Theorems and Probelms in Set Systems

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Let
$$X = [n] = \{1, 2, \ldots, n\}$$
$$\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$$

$X^k$ = the set of all $k$-subsets of $X$.
$\mathcal{F}$ = a family of subsets of $X = [n]$

$\mathcal{F}$ is called $intersecting$ if every pair of distinct subsets $E, F \in \mathcal{F}$ have a nonempty intersection.

$\mathcal{F}$ is called $\mathcal{L}$-$intersecting$ if $|E \cap F| \in \mathcal{L}$ for every pair of distinct subsets $E, F \in \mathcal{F}$.

A $k$-uniform intersecting family = an $\mathcal{L}$-intersecting family for $\mathcal{L} = \{1, 2, \ldots, k-1\}$.

$\mathcal{F}$ is $k$-$uniform$ if it is a collection of $k$-subsets of $X$.

**Theorem 1** (Erdös, Ko, and Rado, 1961).
Let $n \geq 2k$ and let $\mathcal{F}$ be a $k$-uniform intersecting family of subsets of $[n]$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.

**Theorem 2** (Frankl, 1976).
Let $k \geq 2$, $d \geq 2$, and $n \geq dk/(d-1)$. Suppose that $\mathcal{F} \subseteq [n]^k$ such that every $d$ sets of $\mathcal{F}$ have a nonempty intersection, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.

A $d$-*dimensional simplex* is defined as a collection of $d+1$ sets $A_1, A_2, \ldots, A_{d+1}$ such that every $d$ of them have a nonempty intersection, but $A_1 \cap A_2 \cap \cdots \cap A_{d+1} = \emptyset$.

A $2$-dimensional simplex is called a *triangle*.

**Example.** $A_1 = \{1,2\}, A_2 = \{2,3\}, A_3 = \{1,3\}$

**Conjecture 3** (Erdös, 1971)
For $n \geq \frac{3k}{2}$, if $\mathcal{F} \subseteq [n]^k$ contains no triangle, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.

**Conjecture 4** (Chvatal, 1974)
Let $k \geq d+1 \geq 3$, $n \geq k(d+1)/d$, and $\mathcal{F} \subseteq [n]^k$. If $\mathcal{F}$ contains no $d$-dimensional simplex, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.

**Theorem 5** (Chvatal, 1974).
For $n \geq k+2 \geq 5$, if $\mathcal{F} \subseteq [n]^k$ contains no $(k-1)$-dimensional simplices, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.

**Theorem 6** (Frankl and Füredi, 1987).
For $k \geq d+2 \geq 4$, there exists $n_0$ such that for $n > n_0$, if $\mathcal{F} \subseteq [n]^k$ contains no $d$-dimensional simplices, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.

**Theorem 7** (Mubayi and Verstraete, 2005)
For $n \geq \frac{3k}{2}$, if $\mathcal{F} \subseteq [n]^k$ contains no triangle, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.

**Theorem 8** (de Bruijin and Erdös, 1948). If $\mathcal{F}$ is a family of subsets of $X$ satisfing $|E \cap F| = 1$ for every pair of distinct subsets $E, F \in \mathcal{F}$, then $|\mathcal{F}| \leq n$.

**Theorem 9** (Bose, 1949). If $\mathcal{F}$ is a family of subsets of $X$ satisfing $|E \cap F| = \lambda$ for every pair of distinct subsets $E, F \in \mathcal{F}$, then $|\mathcal{F}| \leq n$.

**Theorem 10** (Ray-Chaudhuri and Wilson, 1975).
Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of $s$ nonnegative integers. If $\mathcal{F}$ is a $k$-uniform $\mathcal{L}$-intersecting family of subsets of $X$, then $|\mathcal{F}| \leq \binom{n}{s}$.

**Example.** Let $L = \{0, 1, 2, \ldots, s-1\}$ and $\mathcal{F}$ be the set of all $s$-subsets of $X$.

**Theorem 11** (Frankl and Wilson, 1981).
Let $\mathcal{L} = \{l_1, l_2, \dots, l_s\}$ be a set of $s$ nonnegative integers. If $\mathcal{F}$ is an $\mathcal{L}$-intersecting family of subsets of $X$, then

$$|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

**Example.** Let $L = \{0, 1, 2, \dots, s-1\}$ and $\mathcal{F}$ be the set of all $s$-subsets of $X$.

**Proof.** Let $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ with

$$|F_1| \leq |F_2| \leq \cdots \leq |F_m|.$$

Let $v_i$ be the characteristic vector for $F_i$.

Then $$v_i \cdot v_j = |F_i \cap F_j|.$$

For each $F_i \in \mathcal{F}$, define a polynomial

$$f_i(x) = \prod_{l_j < |F_i|} (v_i \cdot x - l_j).$$

Then $f_i$, $1 \leq i \leq m$, are linearly independent multilinear polynomials of degree at most $s$. Thus

$$|\mathcal{F}| = m \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

**Theorem 12** (Alon, Babai, and Suzuki, 1991).
Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of $s$ nonnegative integers and $K = \{k_1, k_2, \ldots, k_r\}$ be a set of integers satisfying $k_i > s - r$ for every $i$. Let $\mathcal{F}$ be an $\mathcal{L}$-intersecting family of subsets of $X$ such that $|F| \in K$ for every $F \in \mathcal{F}$. Then

$$|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}.$$

**Example.** Let $L = \{0, 1, 2, \ldots, s-1\}$ and $\mathcal{F}$ be the set of all subsets of $X$ with size at least $s - r + 1$ and at most $s$.

**Conjecture 13** (Frankl-Füredi Conjecture, 1984)
Let $\mathcal{L} = \{1, 2, \ldots, s\}$. If $\mathcal{F}$ is an $\mathcal{L}$-intersecting family of subsets of $X$, then $|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}$.

The conjecture was proved by G. Ramanan in 1997.

**Conjecture 14** (Snevily, 1994). Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of $s$ positive integers. If $\mathcal{F}$ is an $\mathcal{L}$-intersecting family of subsets of $X$, then $|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}$.

The conjecture was proved by H. Snevily in 2003.

**Theorem 15** (Chen and Liu). Let $\mathcal{L} = \{1, 2, \ldots, s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be a set of integers satisfying $k_i > s - r$ for every $i$. Let $\mathcal{F}$ be an $\mathcal{L}$-intersecting family of subsets of $X$ such that $|F| \in K$ for every $F \in \mathcal{F}$. Then

$$|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r+1}.$$

By taking $r = n$, we obtain the following:

**Frankl-Füredi Conjecture, 1984**
Let $\mathcal{L} = \{1, 2, \ldots, s\}$. If $\mathcal{F}$ is an $\mathcal{L}$-intersecting family of subsets of $X$, then $|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}$.

**Corollary 16.** Let $\mathcal{L} = \{1, 2, \ldots, s\}$. If $\mathcal{F}$ is a $k$-uniform $\mathcal{L}$-intersecting family of subsets of $X$, then

$$|\mathcal{F}| \leq \binom{n-1}{s}.$$

a $k$-uniform intersecting family = an $\mathcal{L}$-intersecting family for $\mathcal{L} = \{1, 2, \ldots, k-1\}$.

By taking $s = k - 1$, Corollary 16 implies:

**Erdös-Ko-Rado Theorem.** Let $n \geq 2k$ and let $\mathcal{F}$ be a $k$-uniform intersecting family of subsets of $[n]$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.

A family $\mathcal{F}$ of subsets of $[n]$ is called $t$-wise $\mathcal{L}$-intersecting if $|F_1 \cap F_2 \cap \cdots \cap F_t| \in \mathcal{L}$ for any $t$ distinct members $F_1, F_2, \cdots, F_t \in \mathcal{F}$.

**Theorem 17.** (Furedi and Sudakov, 2004)
Let $\mathcal{L} = \{\lambda\}$ and let $3 \leq t \leq n$. If $\mathcal{F}$ is $t$-wise $\mathcal{L}$-intersecting family of subsets of $[n]$, then $|\mathcal{F}| \leq \frac{k}{2}n + 1$.

**Problem 18.** Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of $s$ nonnegative integers and $t \leq s + 2$. If $\mathcal{F}$ is a $k$-uniform $t$-wise $\mathcal{L}$-intersecting family of subsets of $[n]$, then $|\mathcal{F}| \leq \binom{n}{s}$.

# Thank You !