NP-completeness of 4-incidence colorability of a semi-cubic graph

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Definitions and Notations

Let $G = (V, E)$ be a graph. Let

$$I(G) = \{(v, e) : v \in V, e \in E, \text{ and } v \text{ is incident with } e\}$$

be the set of all *incidences* of G. We say that two incidences (v, e) and (w, f) are *adjacent* if one of the following holds :

(1) $v = w$;

(2) $e = f$; *and*

(3) *the edge* vw *equals to* e *or* f.

Definitions and Notations

we view G as a digraph by splitting each edge uv into two opposite arcs (u, v) and (v, u). For $e = uv$, we identify the incidence (u, e) with the arc (u, v). By a slight abuse of notation we will refer to the incidence (u, v) whenever it is convenient to do so.

Two distinct incidences (u, v) and (x, y) are *adjacent* if one of the following holds :

(i) $u = x$;

(ii) $u = y$ *and* $v = x$;

(iii) $v = x$.

Definitions and Notations

The configurations associated with (i)-(iii) are pictured in Fig.1.

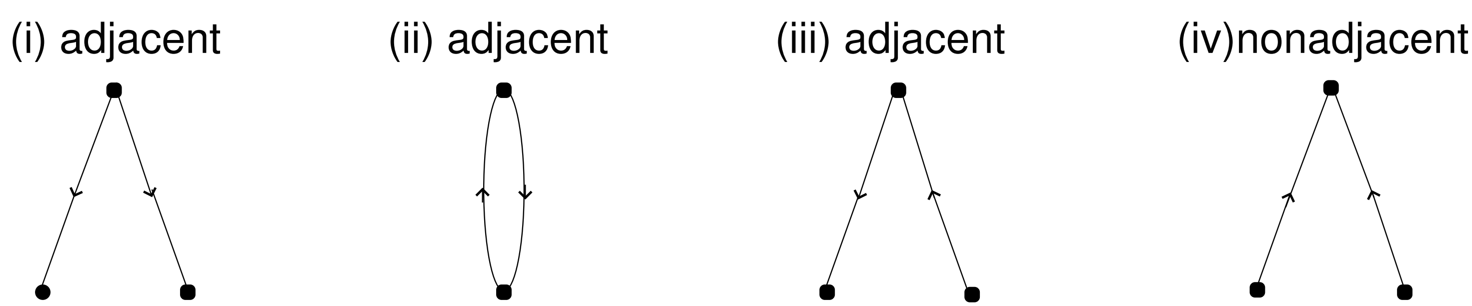


FIG. 1 – Examples of adjacent and nonadjacent incidences.

Introduction

This concept was first developed by Brualdi and Massey in 1993. They posed the incidence coloring conjecture (ICC), which states that for every graph G, $\chi_i(G) \leq \mathit{\Delta} + 2$, where $\mathit{\Delta} = \mathit{\Delta}(G)$ is the maximum degree of G. In 1997, Guiduli pointed out that the ICC was solved in the negative following an example in Alon. He showed that

$$\mathit{\Delta} + \Omega(log\mathit{\Delta}) \leq \chi_i(G) \leq \mathit{\Delta} + O(log\mathit{\Delta})$$

where $\Omega = \frac{1}{8} - o(1)$.

Some results

Theorem 1 *(Brualdi and Massey) For every graph G, $\Delta(G) + 1 \leq \chi_i(G) \leq 2\Delta(G)$.*

Shiu et al. proved that $\chi_i(G) \leq 5$ for several classes of cubic (3-regular) 2-connected graphs G, including Hamiltonian cubic graphs. In 2005 Maydanskiy proved that the conjecture (ICC) holds for all graphs with $\Delta \leq 3$.

Theorem 2 *(Maydanskiy) For every graph G with $\Delta(G) \leq 3$, $\chi_i(G) \leq 5$.*

Main results

Definition 1 *For a graph G with $\Delta = 3$, if the degree of any vertex of G is 1 or 3, then the graph G is called a semi-cubic graph.*

We have the following corollary.

Corollary 3 *The incidence coloring number of a semi-cubic graph is 4 or 5.*

Main results

In this paper, we show the following result.

Theorem 4 *It is NP-complete to determine if a semi-cubic graph is 4-incidence colorable.*

Theorem 5 *It is NP-complete to determine if a general graph is k-incidence colorable, or in other words, it is NP-complete to determine the incidence coloring number for a general graph.*

Incidence coloring of semi-cubic graphs

We define a *strong vertex coloring* of G to be a proper vertex coloring such that if $u, w \in N_G(v)$, then u, w are assigned distinct colors. If $\sigma : V(G) \to C_s$ is a strong vertex coloring of G and $k = |C_s|$, then we say that G is *k-strong-vertex colorable* and σ is a *k-strong-vertex coloring* of G, where C_s is a color-set. And we say that G is k-strong-vertex colored.

Lemma 1 . *Given a semi-cubic graph G, G is 4-incidence colorable if and only if G is 4-strong-vertex colorable.*

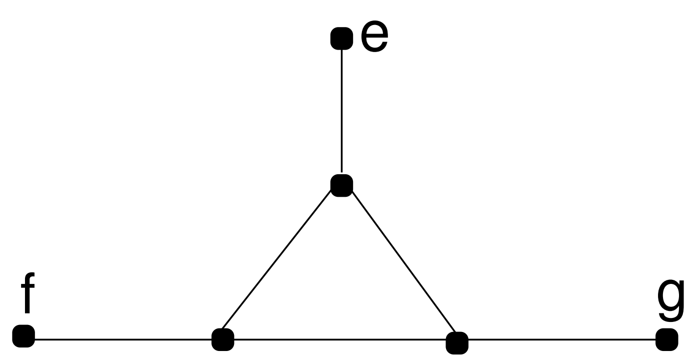


FIG. 2 – The Kite graph

The blocks used in the construction

The $3SAT$ problem is stated as follows :

<u>$3SAT$</u>

INSTANCE : Set U of variables, collection $\mathcal{C}$ of clauses over U such that each clause $C_i \in \mathcal{C}$ has $|C_i| = 3$.

QUESTION : Is there a truth assignment for U such that every $C_i \in \mathcal{C}$ is true ?

The blocks used in the construction

Given an instance $\mathcal{C}$ of the problem 3SAT, we will show how to construct a semi-cubic graph G of polynomial size in terms of the size of the instance $\mathcal{C}$ such that G is 4-strong-vertex colorable if and only if $\mathcal{C}$ is satisfiable, which, from Lemma 1, implies that G is 4-incidence colorable if and only if $\mathcal{C}$ is satisfiable.

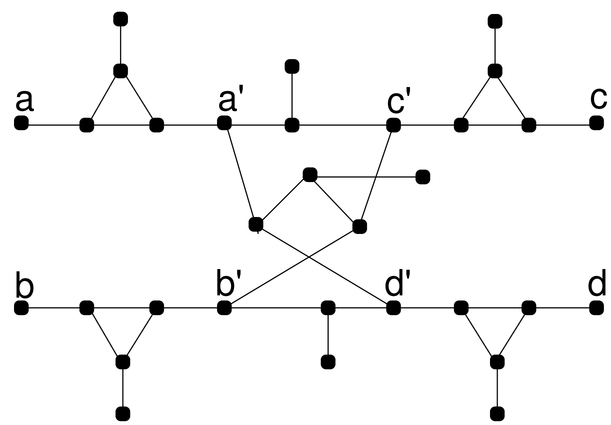
The blocks used in the construction

The semi-cubic graph G will be constructed from some pieces or “blocks" which carry out specific tasks. Information will be carried between blocks by pairs of vertices. In a 4-strong-vertex coloring of the graph G, such a pair of vertices is said to represent the value T (“true") if the vertices have the same color, and to represent F (“false") if the vertices have distinct colors. In the following we always use $S = \{1, 2, 3, 4\}$ to denote the set of colors.

The switch blocks

The inverting component is shown with its symbol in Fig.3. It may be checked that this component is 4-strong-vertex colorable. If this component is 4-strong-vertex colored, one of the pairs of vertices marked a, b or c, d must have equal colors and the remaining pair of vertices must have distinct colors.

The switch bolocks



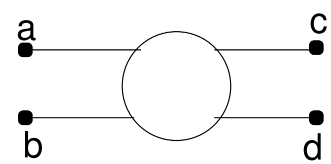


FIG. 3 – The switch block and its symbolic representation.

Regarding the pair of vertices a, b as the input and the pair c, d as the output, the block changes a representation of T to a representation of F, and vice versa.

The switch blocks

In fact, there are only two non-equivalence ways to color the two pairs of vertices a, b and c, d. It is shown in Fig.4.

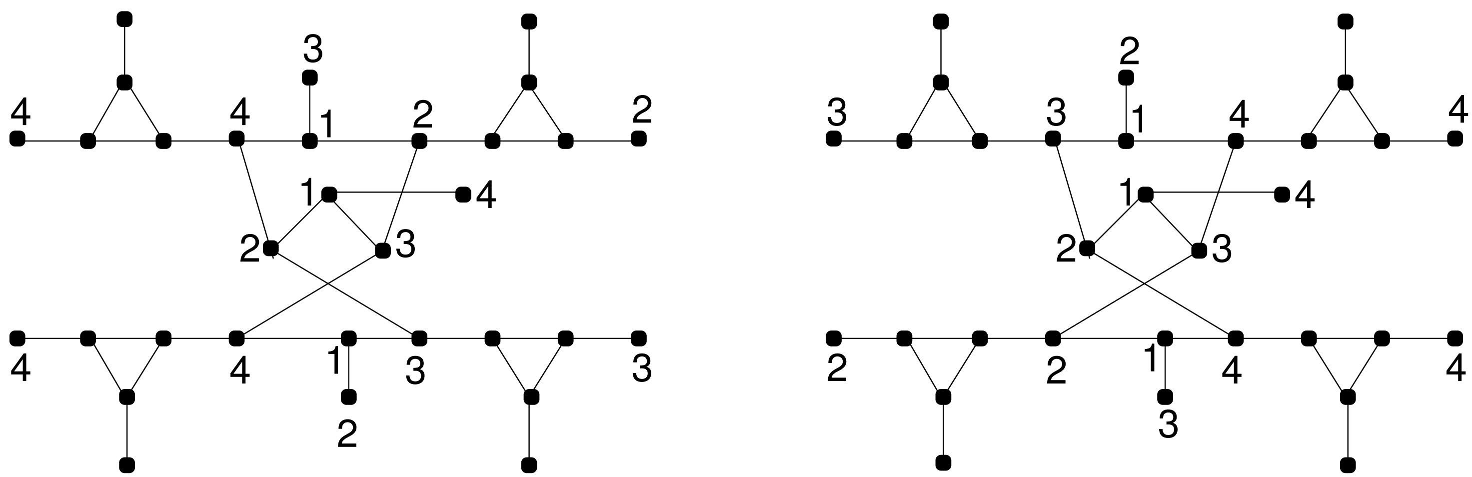


FIG. 4 – Two non-equivalence ways to color the two pairs of vertices a, b and c, d.

The variable blocks

We construct the big switch block using three switch blocks by identifying the output of the first switch block with the input of the second switch block, identifying the output of the second switch block with the input of the third switch block. It is shown with its symbol in Figure 5.

The variable blocks

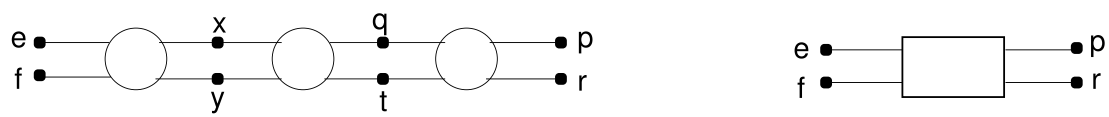


Fig. 5 – The big switch block and its symbolic representation.

Regarding the pair of vertices e, f as the input and the pair p, r as the output of the big switch block.

The variable blocks

Lemma 2 *Given a big switch block G for which the input and output are e, f and p, r, respectively, G is 4-strong-vertex colorable. And if G is 4-strong-vertex colored, one of the following holds :*

1 *if $c(e) = c(f)$, then $c(p) \neq c(r)$ and $c(p)$,$c(r)$ may be any two different colors in $S = \{1, 2, 3, 4\}$;*

2 *if $c(e) \neq c(f)$, then $c(p) = c(r)$ and $c(p)$ may be any color in $S = \{1, 2, 3, 4\}$,*

where $c(v)$ denotes the color assigned to the vertex v of G.

The variable blocks

The truth or falsity of each variable u_i will be represented by a variable block as shown in Figure 6, in which the blocks have, respectively, $1, 2, \cdots, 6$ pairs of output vertices. In general, the number of output pairs in the block representing u_i should be equal to the number of total appearances of u_i or $\overline{u_i}$ among the clauses of $\mathcal{C}$. If k pairs of output vertices is needed, we construct a variable block which is made from k big switch blocks and $2 \cdot \lfloor \frac{k-1}{2} \rfloor$ Kite graphs H.

The variable blocks

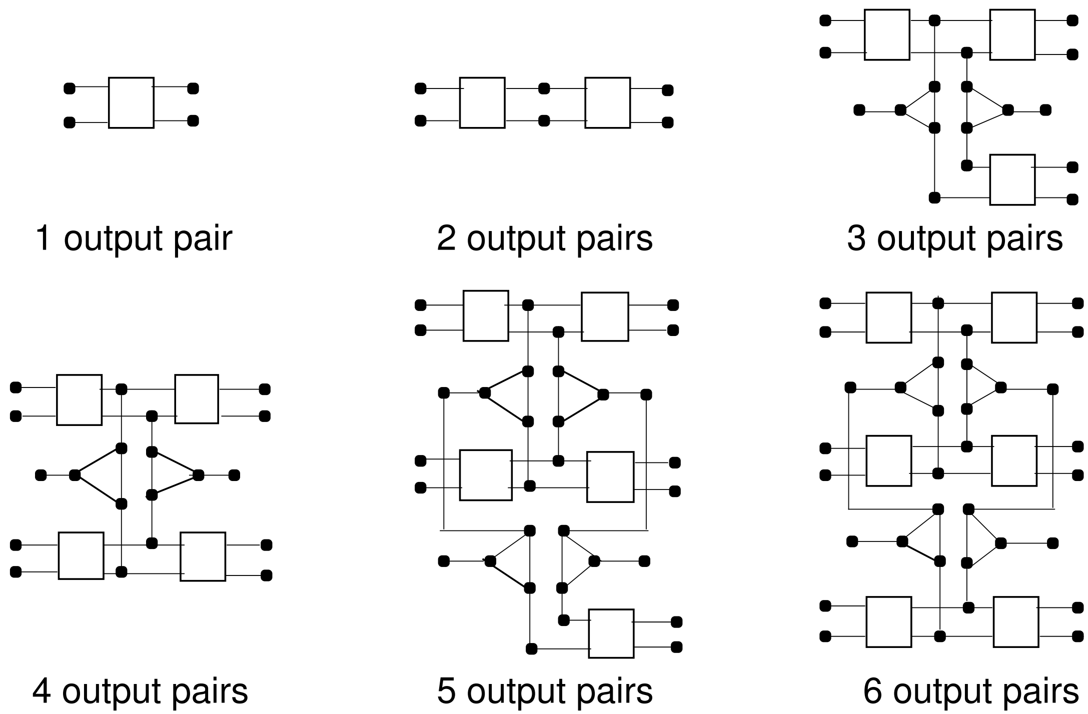


FIG. 6 – The variable blocks having $1, 2, \cdots, 6$ output pairs of vertices respectively.

The variable blocks

Lemma 3 *Given a variable block G, G is 4-strong-vertex colorable. And in any 4-strong-vertex coloring of G, all the output pairs must represent the same value. If the output pairs represent T ("true"), then the color-set of any output pairs may be any color in $S = \{1, 2, 3, 4\}$. If, on the other hand, the output pairs represent F ("false"), then the color-set of any output pairs may be any two different colors in $S = \{1, 2, 3, 4\}$.* ∎

The clause blocks

The truth of each clause C_j will be tested by a clause block as shown in Figure 7. The block is constructed from 3 switch blocks and a cycle of length 10.

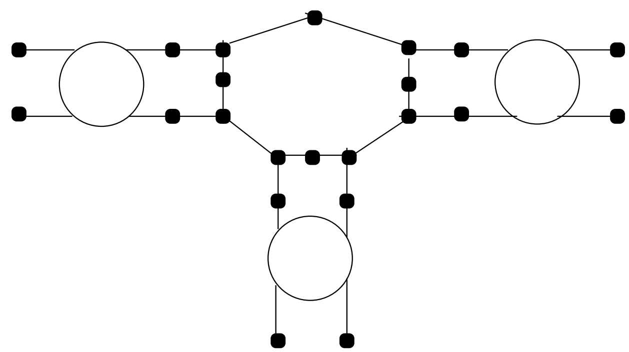


Fig. 7 – The clause block made from 3 switch blocks and a cycle of length 10 and having 3 input pairs of vertices.

The clause blocks

The following lemma is crucial for proving our main theorem.

Lemma 4 *The clause block is 4-strong-vertex colorable if and only if the three input pairs of vertices do not all represent F.*

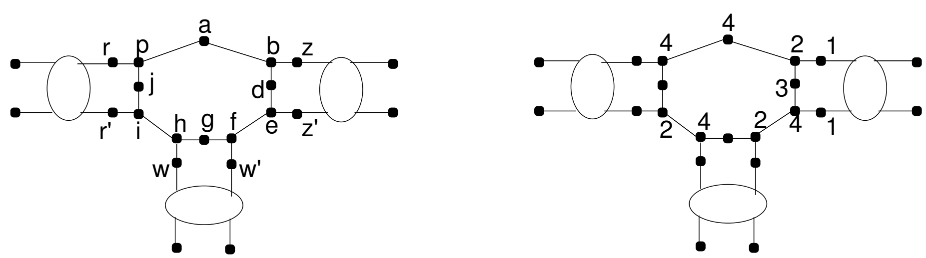


Fig. 8 – The three input pairs of vertices all represent F.

The clause blocks

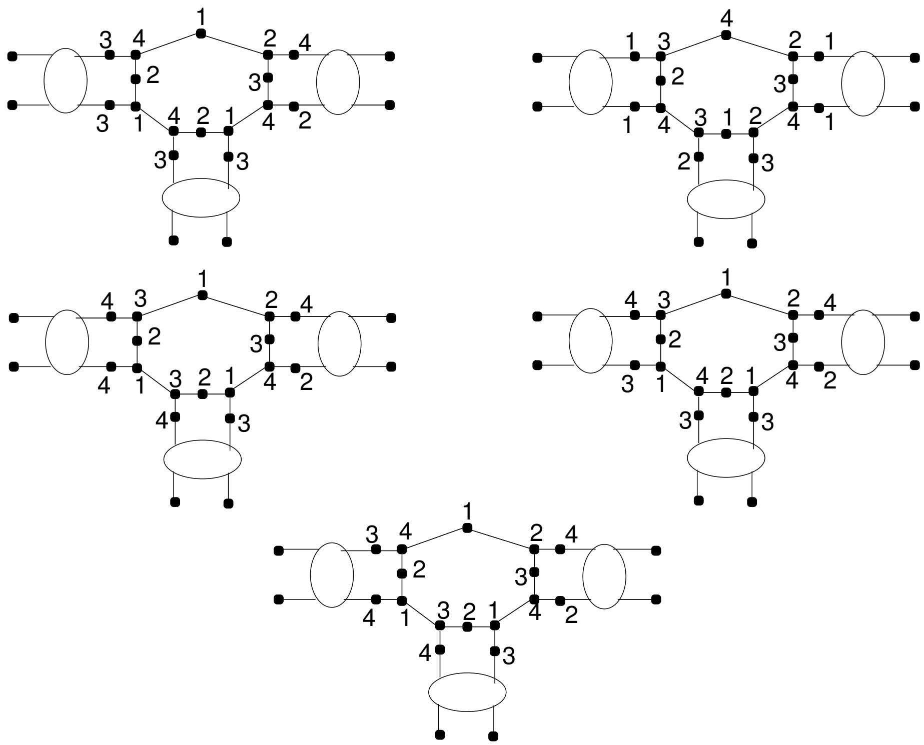


FIG. 9 – The three input pairs of vertices do not all represent F.

Example

Let $\mathcal{C} = \{C_1, C_2\}$ an instance of the problem 3SAT and

$$C_1 = u_1 \vee \overline{u_2} \vee u_3,$$

$$C_2 = u_2 \vee u_3 \vee \overline{u_4}.$$

By the proof of Theorem 4, we can construct the graph G' as shown in Figure 10 and obtain the semi-cubic G by adding a pendant edge to every 2-vertex of G'.

Example

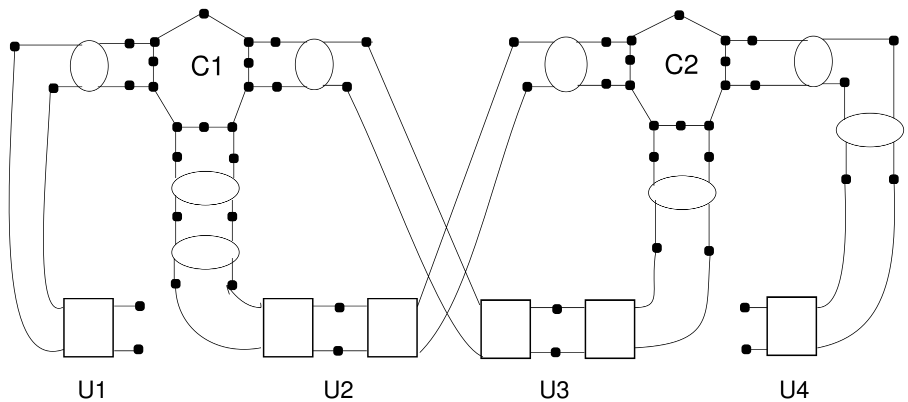


FIG. 10 – The graph G'. We can obtain a semi-cubic graph G by adding a pendant edge to every 2-vertex of G'.