

Spanning Eulerian Subgraphs in claw-free graphs

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GRAPHS ARE FINITE AND LOOPLESS.

A GRAPH IS CLAW-FREE IF IT HAS NO INDUCED SUBGRAPH $\cong K_{1,3}$.

A CONNECTED GRAPH G IS **EULERIAN** IF EVERY VERTEX OF G HAS EVEN DEGREE.

A EULERIAN SUBGRAPH H OF G IS A **DOMINATING EULERIAN SUBGRAPH** OF G IF EACH EDGE OF G HAS AT LEAST ONE END ON H .

A GRAPH G IS **SUPEREULERIAN** IF G HAS A SPANNING EULERIAN SUBGRAPH.

A GRAPH IS **HAMILTONIAN** IF IT HAS A SPANNING EULERIAN SUBGRAPH H WITH $\Delta(H) = 2$.

LET G BE A GRAPH WITH $E(G) \neq \emptyset$. THE LINE GRAPH OF G , WRITTEN $L(G)$, HAS $E(G)$ AS ITS VERTEX SET, WHERE TWO VERTICES ARE ADJACENT IN $L(G)$ IF AND ONLY IF THE CORRESPONDING EDGES ARE ADJACENT IN G .



THEOREM A (Harary and Nash-Williams 1965).

$L(G)$ IS HAMILTONIAN $\iff G$ HAS A DOMINATING EULERIAN SUBGRAPH OR G IS ISOMORPHIC TO $K_{1,s}$ FOR SOME $s \geq 3$.

CONJECTURE B (Matthews and Sumner (1984)).

EVERY 4-CONNECTED CLAW FREE GRAPH IS HAMILTONIAN.

CONJECTURE C (Thomassen (1986)).

EVERY 4-CONNECTED LINE GRAPH IS HAMILTONIAN.

THEOREM D (Ryjáček (1997)). THESE TWO CONJECTURES ARE EQUIVALENT.

MAIN RESULT

A GRAPH G IS ESSENTIALLY K -EDGE-CONNECTED IF FOR ANY $X \subseteq V(G)$ WITH $|X| < K$, AT MOST ONE COMPONENT OF $G - X$ HAS EDGES.

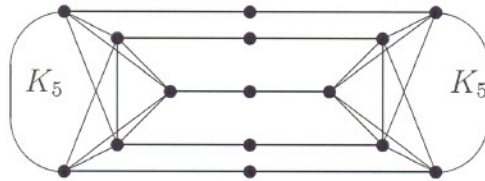


Figure 1

THEOREM 1. IF G IS AN ESSENTIAL 4-EDGE-CONNECTED CLAW-FREE SIMPLE GRAPH WITH $\delta(G) \geq 3$. THEN G HAS A SPANNING EULERIAN SUBGRAPH H WITH $\Delta(H) \leq 4$.

THEOREM 2. IF G IS A CLAW-FREE SUPEREULERIAN GRAPH THEN G HAS A SPANNING EULERIAN SUBGRAPH H WITH $\Delta(H) \leq 4$.

COLLAPSIBLE GRAPHS

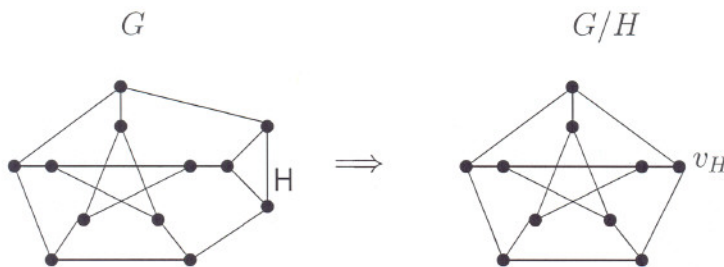
FOR A GRAPH G , LET $O(G)$ DENOTE THE SET OF VERTICES OF ODD DEGREE IN G . A GRAPH G IS CALLED COLLAPSIBLE IF FOR EVERY EVEN SUBSET X OF $V(G)$ THERE EXISTS A SPANNING CONNECTED SUBGRAPH H_X OF G SUCH THAT $O(H_X) = X$.

LET G BE COLLAPSIBLE AND LET u, v BE TWO VERTICES OF G . LET $X = \{u, v\}$ IF $u \neq v$, AND $X = \emptyset$, IF $u = v$. SINCE G IS COLLAPSIBLE, G HAS A SPANNING CONNECTED SUBGRAPH H_X SUCH THAT $O(H_X) = X$. THEN

IF $X = \emptyset$, THEN H_X IS AN SPANNING EULERIAN SUBGRAPH.

IF $X = \{u, v\}$, THEN H_X HAS AN EULERIAN TRAIL STARTING FROM u AND ENDING AT v .

IF H IS A CONNECTED SUBGRAPH OF G , THEN THE CONTRACTION G/H IS OBTAINED FROM G BY CONTRACTING ALL EDGES OF H AND BY DELETING ANY RESULTING LOOPS.



EXAMPLES OF COLLAPSIBLE GRAPHS:

1. C_2 AND $K_3 = C_3$ (cycle of length 3) ARE COLLAPSIBLE.
2. THE FOLLOWING TWO GRAPHS ARE COLLAPSIBLE.

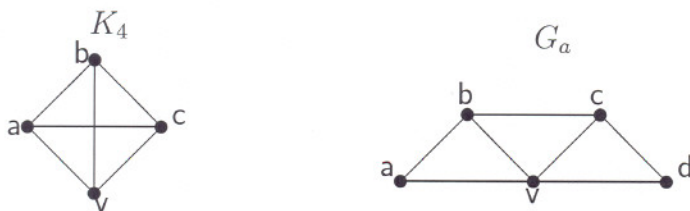


Figure 2

3. ALL COMPLETE GRAPHS K_n OF ORDER $n \geq 3$ ARE IN \mathcal{C} .
4. $K_{3,3} - e$ IS COLLAPSIBLE, where e is an edge in $K_{3,3}$.
5. GRAPHS IN FIGURE 3 ARE NOT COLLAPSIBLE, AND HAVE NO NON-TRIVIAL COLLAPSIBLE SUBGRAPHS.

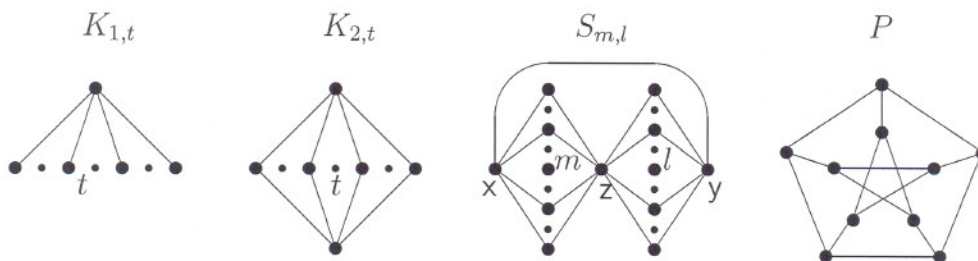
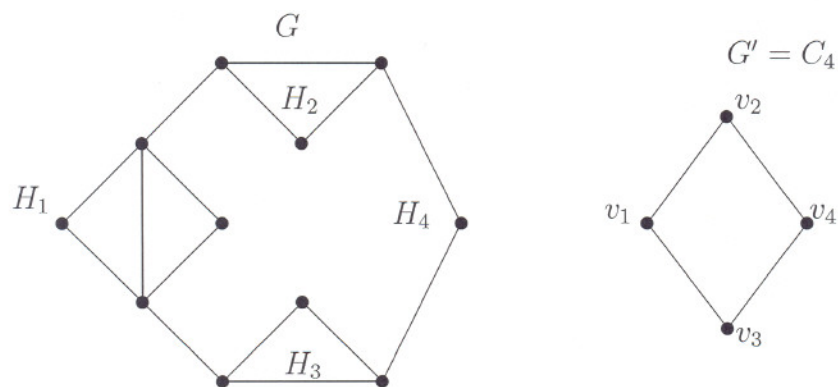
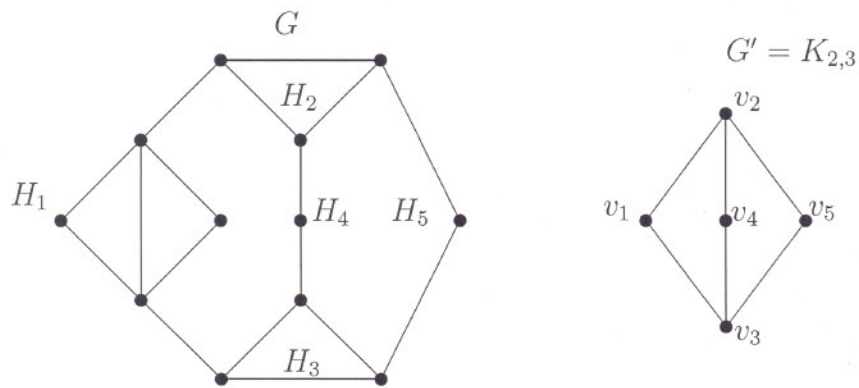


Figure 3

THEOREM E (CATLIN) LET G BE A GRAPH. LET H BE A COLLAPSIBLE SUBGRAPH OF G , THEN G HAS A SPANNING EULERIAN SUBGRAPH IF AND ONLY IF G/H HAS A SPANNING EULERIAN SUBGRAPH.

ANY GRAPH HAS A UNIQUE SET OF MAXIMAL COLLAPSIBLE SUBGRAPHS (Catlin). LET H_1, H_2, \dots, H_c BE THE MAXIMAL COLLAPSIBLE SUBGRAPHS OF G . THE REDUCTION OF G IS THE GRAPH G' OBTAINED FROM G BY CONTRACTING EACH H_i TO A DISTINCT VERTEX v_i . A GRAPH G IS CALLED **REDUCED** IF G IS A REDUCTION OF SOME GRAPH.



THEOREM F (Catlin 1988). LET G BE A GRAPH AND LET G' BE THE REDUCTION OF G . THEN EACH OF THE FOLLOWING HOLDS:

1. G IS COLLAPSIBLE IF AND ONLY IF $G' = K_1$.
2. G HAS A SPANNING EULERIAN SUBGRAPH IF AND ONLY IF G' HAS A SPANNING EULERIAN SUBGRAPH.
3. IF $G' \neq K_1$, G' IS K_3 -FREE AND WITH $\delta(G') \leq 3$.

MAIN RESULT. LET G BE AN ESSENTIAL 4-EDGE-CONNECTED CLAW-FREE SIMPLE GRAPH WITH $\delta(G) \geq 3$. THEN

1. G IS COLLAPSIBLE, AND
2. G HAS A SPANNING EULERIAN SUBGRAPH H WITH $\Delta(H) \leq 4$.

PROOF (OUTLINE). PROVE (1) FIRST

BY WAY OF CONTRADICTION, SUPPOSE G IS NOT COLLAPSIBLE. THEN THE REDUCTION $G' \neq K_1$ WITH $\delta(G') \leq 3$. SINCE G IS ESSENTIALLY 4-EDGE-CONNECTED AND $\delta(G) \geq 3$, $\delta(G') = 3$. LET v BE A VERTEX IN G' WITH DEGREE 3. LET H BE THE PREIMAGE OF v IN G . THEN $|V(H)| = 1$. SINCE G IS CLAW-FREE, TWO OF THE THREE EDGES INCIDENT WITH v WILL BE IN A TRIANGLE, A CONTRADICTION, AND SO G IS COLLAPSIBLE.

PROVE (2):

SINCE G IS COLLAPSIBLE, G HAS SPANNING EULERIAN SUBGRAPHS. LET H BE A SPANNING EULERIAN SUBGRAPH OF G WITH THE EDGE SET $|E(H)|$ AS SMALL AS POSSIBLE. THEN $\Delta(H) \leq 4$.

BY WAY OF CONTRADICTION, SUPPOSE THAT THERE IS A VERTEX v IN $V(H)$ SUCH THAT $d_H(v) = \Delta(H) \geq 6$. THEN WE SHOW THAT WE CAN CONSTRUCT A SPANNING EULERIAN SUBGRAPH FROM H WITH FEWER EDGES. THERE ARE THREE CASES HERE BY CONSIDERING THE COMPONENTS OF $H - v$.

CASE 1. ONE COMPONENT OF $H - v$ HAS AT LEAST SIX VERTICES ADJACENT TO v IN H .

FOR THIS CASE, WE FIRST SHOW THAT H CONTAINS A SUBGRAPH ISOMORPHIC TO K_4 OR G_a .

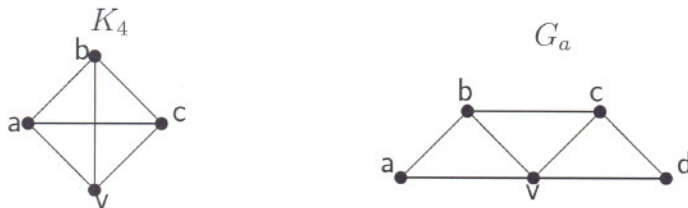


Figure 4

THEN $H_v = H - \{vb, vc, bc\}$ IS A SPANNING EULERIAN SUBGRAPH WITH FEWER EDGES THAN H .

CASE 2. ONE COMPONENT OF $H - v$ HAS FOUR VERTICES ADJACENT TO v IN H .

NOTE THAT v IS A CUT VERTEX IN THIS AND NEXT CASE. SINCE $d_H(v) \geq 6$ ~~6~~ ^{other} THERE ARE AT LEAST TWO VERTICES, SAY u_1 AND u_2 , ADJACENT TO v IN H . LET $\{v_1, v_2, v_3, v_4\}$ BE THE SET OF THE FOUR VERTICES ADJACENT TO v IN H .

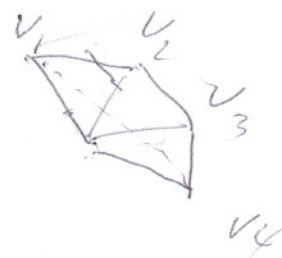
IF THERE ARE TWO VERTICES, SAY v_1 AND v_2 , ARE NOT ADJACENT TO EACH OTHER, THEN WE SHOW THAT EITHER $H_0 = H + u_1v_1 - \{vu_1, vv_1\}$ OR $H_0 = H + u_1v_2 - \{vu_1, vv_2\}$ IS A SPANNING EULERIAN SUBGRAPH OF G WITH FEWER EDGES THAN H .

IF ANY TWO VERTICES IN $\{v_1, v_2, v_3, v_4\}$ ARE ADJACENT TO EACH OTHER IN H , THEN $H_0 = H - \{vv_1, vv_2, v_1v_2\}$ IS A SPANNING EULERIAN SUBGRAPH OF G WITH FEWER EDGES THAN H .

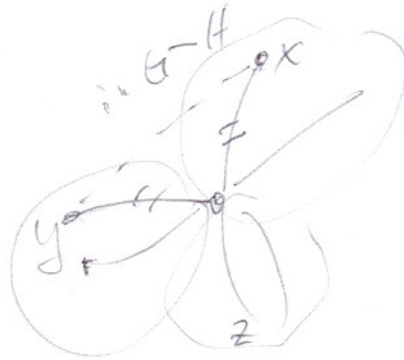
Sub 1



Sub 2



CASE 3. EACH COMPONENT OF $H - v$ HAS EXACTLY TWO VERTICES ADJACENT TO v IN H .



THE RESULT IS BEST POSSIBLE IN THE FOLLOWING SENSE: LET G BE A GRAPH OBTAINED FROM A PATH WITH LENGTH $r \geq 3$ BY REPLACING EACH EDGE IN THE PATH BY A COMPLETE GRAPH OF ORDER 4. THEN G IS AN ESSENTIAL 4-EDGE-CONNECTED CLAW FREE GRAPH WITH $\delta(G) \geq 3$. OBVIOUSLY, G HAS NO SPANNING EULERIAN SUBGRAPH H WITH $\Delta(H) = 2$.