Characterization of trees with equal domination parameters

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An area of research in domination of graphs that has received considerable attention is the study of classes of graphs with equal domination parameters.

For any two graph theoretic parameters $\lambda$ and $\mu$, $G$ is called a $(\lambda, \mu)$-graph if $\lambda(G) = \mu(G)$.
Introduction

1. Cockayne, Favaron, Mynhardt, and Puech (Journal of Graph Theory, 2000) characterized the class of \((\gamma, i)\)-trees, that is trees with equal domination and independent domination numbers.

2. Hattingh, and Henning (Journal of Graph Theory, 2000) provided a constructive characterization of trees with equal independent domination and restrained domination numbers, and a constructive characterization of trees with equal independent domination and weak domination numbers is also given.
3. Haynes, Hedetniemi, and Slater (Discrete Math., 2003) provided constructive characterization of those trees with strong equality of domination parameters (those trees with $\gamma(T) \equiv i(T)$, $\gamma(T) \equiv \gamma_t(T)$, $\gamma(T) \equiv \gamma_p(T)$, respectively).

4. Qiao, Kang, Cardei, and Du (Journal of Global Optimization, 2003) provided a constructive characterization of $(\gamma, \gamma_p)$-trees, that is the trees with equal domination and paired-domination numbers.
Main results

1. **Hou Xinmin (Ars Combin., accepted)** Characterize the \((\gamma, \gamma_t)\)-trees, that is the trees with equal domination and total domination numbers.

2. **Hou Xinmin (submitted)** Characterize the \((2\gamma, \gamma_p)\)-trees, that is the trees for which the paired-domination number is twice the domination number.
Definitions

1. A set $S \subseteq V$ is a dominating set of $G$ if every vertex in $V \setminus S$ is adjacent to some vertex in $S$. (That is $N[S] = V$.) The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A dominating set of $G$ of cardinality $\gamma(G)$ is called a $\gamma$-set of $G$.

Let $G = (V, E)$ be a graph without isolated vertices.

2. A set $S \subseteq V$ is a total dominating set of $G$ if every vertex of $V$ is adjacent to some vertex in $S$. (That is $N(S) = V$.) The total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of $G$. A total dominating set of $G$ of cardinality $\gamma_t(G)$ is called a $\gamma_t$-set of $G$. 
3. A set $S \subseteq V$ is a **paired-dominating set** of $G$ if $S$ dominates $V$ and $G[S]$ contains at least one perfect matching. **The paired-domination number** of $G$, denoted by $\gamma_p(G)$, is the minimum cardinality of a paired-dominating set of $G$. (See figure 1)

**Proposition 1.** Let $G$ be a graph without isolated vertices. Then $\gamma(G) \leq \gamma_t(G) \leq \gamma_p(G) \leq 2\gamma(G)$. 
Definitions

Paired-dominating set

Total dominating set

Dominating set

Paired-dominating set

Figure 1
Characterization of $(\gamma, \gamma_t)$-trees

To state the characterization of trees with equal domination and total domination numbers, we introduce five types of operations.

**Type-1 operation:** Attach a path $P_1$ to a vertex $v$ of $T$, where $v$ is in a $\gamma_t$-set of $T$.

**Type-2 operation:** Attach an end vertex of a path $P_2$ to a vertex $v$ of a tree $T$, where $v$ is in a $\gamma_t$-set of $T$ and for every $\gamma$-set $X$ of $T$, there is no vertex $u \in X$ such that $PN(u, X) = \{v\}$ in $T$. 
Type-3 operation: Attach an end vertex of a path $P_5$ to a vertex $v$ of a tree $T$, where $v$ is in a $\gamma_t$-set of $T$ and for every $\gamma$-set $X$ of $T$, there is no vertex $u \in X$ such that $PN(u, X) = \{v\}$ in $T$.

Type-4 operation: Attach a remote vertex of a path $P_4$ to a vertex $v$ of a tree $T$, where $v$ is a vertex such that for every $\gamma$-set $X$ of $T$, there is no vertex $u \in X$ such that $PN(u, X) = \{v\}$ in $T$.

Type-5 operation: Attach a vertex $u_0$ of $T_1$ to a vertex $v$ of a tree $T$, where $T_1$ is a tree with $V(T_1) = \{u_0, u_1, u_2, u_3, u_4\}$ and $E(T_1) = \{u_0u_1, u_1u_2, u_1u_3, u_2u_4\}$. 
Characterization of \((\gamma, \gamma_t)\)-trees

**Figure 2:**
\[ \gamma(P_4) = \gamma_t(P_4) = \gamma_p(P_4) = 2 \quad (b) \]
\[ \gamma(T) = \gamma_t(T) = 3 < \gamma_p(T) = 4. \]
(a) gives the tree \(T\) of minimum order with \(\gamma(T) = \gamma_t(T) = \gamma_p(T)\), (b) gives the tree \(T\) of minimum order with \(\gamma(T) = \gamma_t(T) < \gamma_p(T)\).
Characterization of \((\gamma, \gamma_t)\)-trees

Let

\[ \mathcal{J}_t = \{ T : \gamma(T) = \gamma_t(T) \}. \]

We define the family \(\mathcal{F}_t\) as:

\[ \mathcal{F}_t = \{ T : T \text{ is obtained from } P_4 \text{ by a finite sequence of operations of Type-}i, i = 1, 2, 3, 4, 5 \}. \]

We shall prove that

**Theorem 1.**

\[ \mathcal{J}_t = \mathcal{F}_t. \]

Proof. By induction.
Characterization of \((2\gamma, \gamma_p)\)-trees

To state the characterization of \((2\gamma, \gamma_p)\)-trees, we introduce three types of operations.

**Type-1 operation:** Attach a path \(P_1\) to a vertex \(v\) of a tree \(T\), where \(v\) is in a \(\gamma\)-set of \(T\) and \(v \notin L(T)\).

**Type-2 operation:** Attach a path \(P_2\) to a vertex \(v\) of a tree \(T\), where \(v\) is a vertex such that for every \(\gamma_p\)-set \(S\) of \(T\), \(PN(v, S) = \phi\) and \(PN(\{v, \bar{v}\}, X) \neq \phi\).

**Type-3 operation:** Attach a path \(P_3\) to a vertex \(v\) of a tree \(T\), where either \(v\) is a vertex of a \(\gamma\)-set of \(T\) such that \(v \notin L(T)\) and, for every \(\gamma_p\)-set \(S\) of \(T\), \(PN(\{v, \bar{v}\}, S) \neq \phi\) if \(\bar{v} \notin L(T)\), or \(v\) is a vertex such that for every \(\gamma_p\)-set \(S\) of \(T\), \(\bar{v} \notin N(S \setminus \{v, \bar{v}\})\) if \(PN(\{v, \bar{v}\}, S) = \phi\).
Characterization of \((2\gamma, \gamma_p)\)-trees

Let
\[ \mathcal{J}_p = \{ T : \gamma_p(T) = 2\gamma(T) \}. \]

We define the family \(\mathcal{F}_p\) as:
\[ \mathcal{F}_p = \{ T : T \text{ is obtained from } P_3 \text{ by a finite sequence of operations of Type-1, Type-2 or Type-3} \}. \]

We shall prove that

**Theorem 3.**
\[ \mathcal{J}_p = \mathcal{F}_p \cup \{ P_2 \}. \]

Proof. By induction.
Thank you!