

REDUCTION FOR 3-CONNECTED GRAPHS OF MINIMUM DEGREE AT LEAST FOUR

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Abstract: A reduction theorem is obtained for the class of 3-connected graphs of minimum degree at least four using three basic operations: contraction of an edge, splitting and crown reduction. We prove that if a vertex x in a 3-connected graph of order at least six and minimum degree at least four is specified, then we can always find one of the following in the proximity of x : (1) an edge whose contraction results in a 3-connected graph of minimum degree at least four, (2) a vertex of degree four at which some splitting produces a 3-connected graph of minimum degree at least four, (3) a crown whose reduction gives a 3-connected graph of minimum degree at least four.

1. HISTORY AND MOTIVATION

- W.T. Tutte (1961): 3-connected graphs
- P. Slater (1974); M. Fontet (1978); N. Martinov (1982, 1990); W. Mader (1984); A. Saito (2003): 4-connected graphs
- A large number of literature exists on **inductive** and **recursive** constructions of classes of graphs. This somewhat parallels activity in this direction in mathematical logic.

2. CONCEPTS

Let $G = (V, E)$ be a graph. A **contraction** of G is defined to be a partition (V_1, V_2, \dots, V_s) of V such that for each $i = 1, 2, \dots, s$, the induced subgraph $G|_{V_i}$ is connected. This partition gives rise to a natural mapping from G to a graph H , also called a **contraction** (graph) of G . The contraction (graph) H is the graph with

$$V(H) = \{V_1, V_2, \dots, V_s\}, \quad E(H) = \{V_i V_j : i \neq j, [V_i, V_j] \neq \emptyset\}.$$

The mapping f is called a **contraction** (mapping) from G onto H , and G is said to be **contractible** to H .

Note that a contraction may be defined in many equivalent ways.

For a graph G and $e = uv \in E(G)$, we denote by G/e the graph obtained from contraction of e . This may be given in terms of the above definition by a graph G/e defined by the partition

$$V(G/e) = \{\{u, v\}, \{u_1\}, \{u_2\}, \dots, \{u_{n-2}\}\}$$

where $V(G) = \{u, v, u_1, \dots, u_{n-2}\}$, and specification

$$E(G/e) = \{[U, V] : U, V \in V(G/e), [U, V]_G \neq \emptyset\}.$$

In this, $[U, V]_G = \{uv : u \in U, v \in V, uv \in E(G)\}$. If the subscript is dropped, $[U, V]$ denotes the set of all edges between U and V . That is

$$[U, V] = \{uv : u \in U, v \in V, u \neq v\}.$$

Let $x \in V(G)$ and $\deg_G x = 4$ with $N(x) = \{x_1, x_2, x_3, x_4\}$. Then the graph G_{xx_1, xx_2} is defined by

$$G_{xx_1, xx_2} = (G - x) \cup \{x_1x_2, x_3x_4\},$$

and the operation to obtain G_{xx_1, xx_2} is called **splitting** at x across $\{x_1, x_2\}$.

Let $S \subset V(G)$. Then a **completion** of S in G is the graph $G \cup [S, S]$.

Let $X = \{x, y, z\}$, $Y = \{u, v\}$ and $X \cap Y \neq \emptyset$. Then the graph C with

$$V(C) = X \cup Y, \quad E(C) = [X, Y] \cup \{uv\}$$

is called a **crown**. A graph G is said to have a crown C if $C \subset G$ and $\deg_G u = \deg_G v = 4$. Note that C is a subgraph of G , but may not be an induced subgraph of G . Though the condition entails $N_G(u) = \{v, x, y, z\}$ and $N_G(v) = \{u, x, y, z\}$, there may be edges of $G - E(C)$ among $\{x, y, z\}$. A **crown reduction** of C is defined to be the operation to obtain

$$H = (G - Y) \cup [X, X].$$

(Equivalently, H is the completion of X in $G - Y$.)

Note that even if G is a 3-connected graph $\delta(G) \geq 4$, and $\deg_G u = \deg_G v = 4$, no splitting at u or v yields a graph H of with $\delta(H) \geq 4$.

Let \mathcal{P} be a graph property. An edge $e \in E(G)$ is **\mathcal{P} -contractible** if $G/e \in \mathcal{P}$. If \mathcal{P} is the class of k -connected graphs, then a \mathcal{P} -contractible edge is often referred to as a **k -contractible** edge. This talk deals with the class $\mathcal{P}_{3,4}$ of 3-connected graphs with $\delta \geq 4$.

3. REDUCTION THEOREMS

The main theorem of this talk.

THEOREM 1 Let G be a 3-connected graph $\delta(G) \geq 4$ and $|G| \geq 6$, and let $x \in V(G)$. Then one of the following holds.

1. There exists an edge $e \in E(G)$ incident with a vertex distance at most two apart from x such that G/e is 3-connected and $\delta(G/e) \geq 4$.
2. There exists a vertex $v \in V(G)$ with $d(v) = 4$ such that the distance between x and v is at most two and there exists a splittings at v yields a 3-connected graph $\delta \geq 4$.
3. There exists a crown $C \subset G$ with $x \in V(C)$ such that crown reduction of G using C yields a 3-connected graph with $\delta \geq 4$.

The proof proceeds by a series of results.

LEMMA 2 Let G be a graph and let $u \in V(G)$.

1. If u belongs to a minimum separator S of G , then $N_G(u) \cap A \neq \emptyset$ for every component A of $G - S$.
2. If $N_G(u)$ induces a complete graph, then u does not belong to any minimum separator.
3. Let H be the graph obtained from $G - u$ by the completion of $N_G(u)$. Then $\kappa(H) \geq \kappa(G)$.

4. CONTRACTIONS

LEMMA 3 Let $k \geq 2$ and let G be a k -connected graph. If $e = xy \in E(G)$ is not k -contractible, then G has a separator S with $|S| = k$ and $\{x, y\} \subset S$.

LEMMA 4 Let G be a 3-connected graph and let $e = xy \in E(G)$. If $G[N_G(x) - \{y\}]$ is 2-connected, then e is 3-contractible.

LEMMA 5 Let G be a 3-connected graph with $\delta(G) \geq 4$. Then for each $x \in V(G)$, there exists a 3-contractible edge incident with x .

LEMMA 6 Let G be a 3-connected graph, and suppose G has a vertex y of degree 4. Let $N_G(y) = \{x, u, v, w\}$. If $xu \in E(G)$, then either yv or yw is 3-contractible.

5. SPLITTING

LEMMA 7 Let G be a 3-connected graph, and let x be a vertex of G of degree four. Let $N_G(x) = \{x_1, x_2, x_3, x_4\}$. Then the following two statements are equivalent.

1. G_{xx_1, xx_2} is not 3-connected.
2. There exists a separator S with $|S| = 3$ and $x \in S$, and G has a fragment A associated with S such that
 - (i) $\{x_1, x_2\} \subset S \cup A$ and $\{x_1, x_2\} \cap A \neq \emptyset$, and
 - (ii) $\{x_3, x_4\} \subset S \cup \bar{A}$ and $\{x_3, x_4\} \cap \bar{A} \neq \emptyset$.

LEMMA 8 Let G be a 3-connected graph, and let x be a vertex of G of degree four. Let $N_G(x) = \{x_1, x_2, x_3, x_4\}$. If $\{x_1x_2, x_3x_4\} \subset E(G)$, then either G_{xx_1, xx_3} or G_{xx_1, xx_4} is 3-connected.

6. CROWN REDUCTION

LEMMA 9 Let G be a 3-connected graph of minimum degree at least four and order at least six. Suppose G has a crown C . Then either

1. crown reduction of C yields a 3-connected graph of minimum degree at least four, or
2. G has an edge e incident with a vertex in C such that G/e is a 3-connected graph of minimum degree at least four.

PROOF OF THEOREM 1

Let G be a 3-connected graph with $\delta(G) \geq 4$ and let $x \in V(G)$. Assume that

- (1) there exists no vertex at distance at most two from x with which a $\mathcal{P}_{3,4}$ -contractible edge is incident,
- (2) there exists no vertex of degree four at distance at most two from x at which any splitting yields a graph in $\mathcal{P}_{3,4}$, and
- (3) G has no crown C containing x such that crown reduction of C yields a graph in $\mathcal{P}_{3,4}$.

LEMMA 10 Graph G has no crown containing x .

LEMMA 11 There exists a vertex y of degree four in $N_G(x)$ such that $G[N_G(y)]$ contains P_4 , in which at least one of the two inner vertices has degree four in G .

A REMARK ABOUT THE CROWN REDUCTION