平面图的边染色、全染色和线性荫度

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本报告涉及如下概念: 边染色(Edge coloring)、全染色(Total coloring)、列表（全）染色(List edge (total) coloring)、线性荫度(Linear arboricity)、线性k-荫度(Linear k-arboricity)等.

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1 Definitions, Results and Conjectures

- The *edge chromatic number* $\chi'(G)$ is the minimum number of colors needed to color the edges of G in such a way that no two adjacent edges are assigned the same color. Vizing's theorem states that for any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$. A graph G is *class one* if $\chi'(G)=\Delta$, and *class two* if $\chi'(G)=\Delta+1$.

  Conjecture A. Every planar graph with $\Delta \geq 6$ is class one.

- The *total chromatic number* of G, $\chi''(G)$, is the smallest integer k such that G has a total k-coloring. Behzad and Vizing posed independently the famous conjecture:

  Conjecture B. For any graph G, $\Delta(G)+1 \leq \chi''(G) \leq \Delta(G)+2$.

- The linear arboricity $la(G)$ of a graph G is the minimum number of linear forests which partition the edges of G. Akiyama, Exoo and Harary conjectured that $la(G) = \lceil \frac{\Delta(G)+1}{2} \rceil$ for any regular graph G. So the conjecture is equivalent to the following conjecture.

  Conjecture C. For any graph G, $\lceil \frac{\Delta(G)}{2} \rceil \leq la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$.

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2 Some analogous results

Theorem 1 *Let G be a planar graph having girth at least g and maximum degree at least Δ. Then*

(1) $\chi'(G) = \Delta(G)$ *if* $(\Delta, g) \in \{(7,3),(5,4),(4,5),(3,8)\}$.

(2) $\chi''(G) \leq \Delta(G) + 2$ *if* $\Delta \neq 6$. *Moreover,* $\chi''(G) = \Delta(G) + 1$ *if* $(\Delta, g) = \{(14,3),(12,3),(11,3),(10,3)$[2006, 吴建良]$,(7,4),(5,5),(4,6),(3,10)\}$.

(3) $la(G) \leq \lceil\frac{\Delta(G)+1}{2}\rceil(\Delta \neq 7$, $\Delta = 7$[2006, 吴建良, 吴玉文]). *Moreover* $la(G) = \lceil\frac{\Delta(G)}{2}\rceil$ *if* $(\Delta, g) = \{(13,3),(11,3),(9,3),(7,4),(5,5),(3,6)\}$.

Similarly, there are some results on a graph which are embedded in a surface of nonnegative characteristic (the plane, projective plane, torus or Klein bottle). Some results are obtained for a graph embedded on a surface of Euler characteristic $\varepsilon < 0$, too. There are more general results.

- Any edge-coloring critical graph of order n and maximum degree $\Delta \geq 8$ has the size at least $3(n + \Delta - 8)$[2002, 苗连英，吴建良].
- Let $mad(G) = \max\{|E(H)|/|V(H)| : H$ is a nonempty subgraph of G $\}$. Then $la(G) = \lceil\Delta(G)/2\rceil$ if one of the following conditions holds: (a) $mad(G) \leq 2$ and $\Delta(G) \geq 7$; (b) $mad(G) \leq 115/64$ and $\Delta(G) \geq 5$; (c) $mad(G) < 3/2$ and $\Delta(G) \geq 3$[2005, 吴建良].

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Theorem 2 *Let G be a connected planar graph such that G has no cycle of length k, where $k \geq 4$. Then*

(1) $\chi'(G) = \Delta(G)$ *if* $(\Delta, k) \in \{(5,4),(5,5)$[2006，吴建良,吴玉文]$,(6,6)\}$;

(2) $\chi''(G) = \Delta + 1$ *if* $\Delta \geq 7$ *and* $4 \leq k \leq 6$[2006，侯建锋等].

(3) $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$ *if* $\Delta \geq 7$ *and* $4 \leq k \leq 6$[2006，吴建良,吴玉文,侯建锋].

Theorem 3 *Let G be a planar graph such that G has no cycle of length from 4 to k, where $k \geq 4$. Then*

(1) $\chi'(G) = \Delta$ *if* $(\Delta, k) \in \{(6,4),(5,5),(4,14)\}$;

(2) $\chi''(G) = \Delta + 1$ *if* $(\Delta, k) \in \{(7,4),(6,5),(5,7),(4,14)\}$.

此外，孙向勇等还考虑了任何两个3圈不共边或不共点的平面图的边染色、全染色和线性荫度，得到了一些类似的结果。当然大家可以考虑另外一些特殊的平面图(如Hamilton平面图)的染色问题.

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A graph is a *series-parallel graph* (K_4-*minor-free graph*)if it contains no subgraphs homeomorphic to K_4. Duffin showed that a 2-connected graph can be obtained from a K_2 by repeatedly applying the following operation: inserting a vertex into an edge (series) or duplicating an edge by a path of length 2 (parallel).

Theorem 4 *Let G be a series-parallel graph. Then $\chi'(G) = \Delta(G)$, $\chi''(G) = \Delta(G) + 1$ and $la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ if $\Delta \geq 3$;*

Since an outerplanar graph is also a series-parallel graph, the theorem is true for any outerplanar graph.

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A graph G is said to be *totally* f*- choosable* if, whenever we give lists A_x of $f(x)$ colors to each element $x \in VE(G)$, there exists a proper total coloring of G where each element is colored with a color from its own list. If $f(x) = k$ for every element $x \in VE(G)$, we said G is *totally* k*-choosable*. The *list total chromatic number* $\chi''_{\text{list}}(G)$ is the smallest integer k such that G is totally k-choosable. The *list edge chromatic number* $\chi'_{\text{list}}(G)$ of G is defined similarly in terms of coloring edges alone, as well as the concept of *edge* f*-choosable*.

Conjecture D. For any graph G, (a) $\chi'_{\text{list}}(G) = \chi'(G)$ and (b) $\chi''_{\text{list}}(G) = \chi''(G)$.

The part (a) of the conjecture was independently posed by Vizing, by Gupta, by Abertson and Collins, and by Bollobás and Harris, and is well known as the **list edge coloring conjecture**. Part (b) of the conjecture was posed by Borodin, Kostochka and Woodall [JCT(B),1997].

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The linear k-arboricity $la_k(G)$ of G is the least integer t such that G can be partitioned into t edge-disjoint forests, whose component trees are paths of length at most k. Clearly, $la_k(G) \geq la_{k+1}(G)$ for any $k \geq 1$. For extremities, $la_1(G)$ is the edge chromatic number, or chromatic index, $\chi'(G)$ of G; $la_\infty(G)$ representing the case when component paths have unlimited lengths is the ordinary linear arboricity $la(G)$ of G.

The linear k-arboricity of a graph was first introduced by Habib and Peroche [Some problems about linear arboricity. Discrete Math. **41**, 219-220 (1982)]. They made the following conjecture:

Conjecture E. If G is a graph with n vertices and $k \geq 2$, then

$$la_k(G) \leq \lceil \frac{n\Delta + \alpha}{2\lfloor kn/(k+1) \rfloor} \rceil$$

where $\alpha = 1$ when $\Delta(G) < n - 1$ and $\alpha = 0$ when $\Delta(G) = n - 1$.

- An *edge-face coloring* of a plane graph G is a coloring of $EF(G)$ such that no two adjacent or incident elements receive the same color. The *edge-face chromatic number*, denoted by $\chi_{ef}(G)$, of a plane graph G is the minimal number of colors needed for an edge-face coloring of G. In 1975, Mel'nikov made the following conjecture:

  Conjecture F. $\Delta(G) \leq \chi_{ef}(G) \leq \Delta(G) + 3$ for every plane graph G.

  The conjecture has been proved by Waller [JCT(B),1997], and Wang and Lih [Disc. Math., 2002].

- An *entire coloring* of a plane graph G is a coloring of $V(G) \cup E(G) \cup F(G)$ such that no two adjacent or incident elements receive the same color. The *entire chromatic number*, denoted by $\chi_{vef}(G)$, of G is the minimal number of colors needed for an entire coloring of G. In 1973, Kronk and Mitchem posed the entire coloring conjecture:

  Conjecture G. $\Delta(G) + 1 \leq \chi_{vef}(G) \leq \Delta(G) + 4$ for every plane graph G,

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