Bondage Number and Efficient Domination of Vertex-transitive Graphs

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August 13, 2006

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- Edge Removal: $\gamma(G - e) = \gamma(G)$ or $\gamma(G) + 1$ [1979,1983].

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- Definition [Fink et al., 1990]:
  $b(G) = \min\{|F| : F \subseteq E(G), \gamma(G - F) > \gamma(G)\}$.
- Difficult to determine $b(G)$ in general.

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- Corresponding to perfect error correcting codes [Biggs].

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- $b(G) \leqslant d(x) + d(y) - 1 - |N(x) \cap N(y)|, xy \in E(G).$

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- $b(G) \geqslant s(G)$.

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$$b(G) \geqslant \begin{cases} \lceil n(G)/2\gamma(G) \rceil & \text{if } G \text{ is undirected;} \\ \lceil n(G)/\gamma(G) \rceil & \text{if } G \text{ is directed.} \end{cases}$$

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- $$b(G) \leqslant \begin{cases} k & \text{if G is undirected and } n \geqslant \gamma(k+1) - k + 1; \\ k+1+l & \text{if G is directed and } n \geqslant \gamma(k+1) - l,\ 0 \leqslant l \leqslant k-1. \end{cases}$$

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- If G has an efficient dominating set, then

$$\begin{cases} \lceil \frac{k+1}{2} \rceil \leqslant b(G) \leqslant k & \text{if G is undirected;} \\ b(G) = k+1 & \text{if G is directed.} \end{cases}$$

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- Let $G = \overrightarrow{C}_s \times \overrightarrow{C}_t$. Then $b(G) = 3$ if both s and t are multiples of 3.
- Let $G = \overrightarrow{C}_{(t+1)k_1} \times \overrightarrow{C}_{(t+1)k_2} \times \cdots \overrightarrow{C}_{(t+1)k_t}$ for any positive integers $k_1, k_2, \ldots, k_t$. Then $b(G) = t + 1$.

Other Applications

- If $n = 2^m - 1$ for a natural number m, then $2^{m-1} \leqslant b(Q_n) \leqslant 2^m - 1$.

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- Let $G = CCC(n)$ be the n-dimensional cube-connected cycles with $n \geqslant 3$ and $n \neq 5$. Then $\gamma(G) = 2^{n-2}n$ and $2 \leqslant b(G) \leqslant 3$. In addition, $b(CCC(3)) = b(CCC(4)) = 2$.

Thank You!