Connectivity and Edge-connectivity of Cartesian Products of Graphs

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Definition of Cartesian Product

Definition
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Their Cartesian product, denoted by $G_1 \square G_2$, take $V_1 \times V_2$ as vertex set. Two vertices $x_1x_2$ and $y_1y_2$ in the Cartesian product are adjacent if and only if $x_1 = y_1, x_2y_2 \in E_2$ or $x_2 = y_2, x_1y_1 \in E_1$. 
Sabidussi’s Result

Theorem (Sabidussi, 1957)
Let $G_1$ and $G_2$ be two connected graph, then
\[ \kappa(G_1 \Box G_2) \geq \kappa(G_1) + \kappa(G_2). \]

Corollary (Sabidussi, 1957)
Let $G_1$ and $G_2$ be two maximally connected graph, then
\[ \kappa(G_1 \Box G_2) = \kappa(G_1) + \kappa(G_2). \]

Sabidussi use this result to show that there are infinitely many non-isomorphic graphs with the same given automorphism group and the same connectivity.
Theorem (Xu, 1998)

Let $D_1$ and $D_2$ be two strongly connected digraph, then
\[ \kappa(D_1 \square D_2) \geq \kappa(D_1) + \kappa(D_2). \]

Theorem (Chiue and Shieh, 1999)

Let $G_1$ and $G_2$ be two connected graph, then
\[ \lambda(G_1 \square G_2) \geq \lambda(G_1) + \lambda(G_2). \]

- These recent study are motivated by their applications in the design of interconnection networks.
Theorem (Xu and Yang 2003)

Let $G_1$ and $G_2$ be two nontrivial connected graphs, then

$$\lambda(G_1 \Box G_2) = \min\{v_1\lambda_2, v_2\lambda_1, \delta_1 + \delta_2\},$$

where $v_i$, $\delta_i$ and $\lambda_i$ are the order, the minimum degree and the edge-connectivity of the graph $G_i (i = 1, 2)$, respectively.
$\lambda(G_1 \square G_2) = \min\{v_1 \lambda_2, v_2 \lambda_1, \delta_1 + \delta_2\}$
Proof.

- By Menger’s Theorem, it suffices to show there are 
  \( \min\{\nu_1\lambda_2, \nu_2\lambda_1, \delta_1 + \delta_2\} \) edge-disjoint paths between any pair of vertices.
- Let \( x_1x_2 \) and \( y_1y_2 \) be a pair of vertices, there are 3 cases:
  1. \( x_1 = y_1 \);
  2. \( x_2 = y_2 \);
  3. \( x_1 \neq y_1 \) and \( x_2 \neq y_2 \).
Other Result by Menger Method

\[ \kappa(G_1 \square G_2) \geq \min \{ \kappa_1 + \delta_2, \kappa_2 + \delta_1 \} \]

\[ \kappa(D_1 \square D_2) \geq \min \{ \kappa_1 + \delta_2, \kappa_2 + \delta_1, 2\kappa_1 + \kappa_2, 2\kappa_2 + \kappa_1 \} \]
Theorem (Xu and Yang 2005)

Let $G_1$ and $G_2$ be two nontrivial connected graphs, then

$$\kappa(G_1 \Box G_2) = \min\{v_1 \kappa_2, v_2 \kappa_1, \delta_1 + \delta_2\},$$

where $v_i$, $\delta_i$ and $\kappa_i$ are the order, the minimum degree and the connectivity of graph $G_i (i = 1, 2)$, respectively.
Proof

- Let $S$ be a minimum separating set.
- If $G_2^x - S_x$ is disconnected, then $|S_x| \geq \kappa_2$. 

\[
S_x = S \cap (\{x\} \times V_2)
\]
If $G_2^x - S_x$ is disconnected and $y \in N_{G_1}(x)$, then $|S_x| + |S_y| \geq \delta_2 + 1$ and $|S_y| \geq 1$. 

\[ S_x = S \cap (\{x\} \times V_2) \]
Generalization to Digraphs

Theorem (Xu and Yang 2005)

Let $D_1$ and $D_2$ be two nontrivial strongly connected digraphs, then
\[ \kappa(D_1 \square D_2) = \min\{\delta_1^+ + \delta_2^+, \delta_1^- + \delta_2^-, v_1\kappa_2, v_2\kappa_1\} \]
and
\[ \lambda(D_1 \square D_2) = \min\{\delta_1^+ + \delta_2^+, \delta_1^- + \delta_2^-, v_1\lambda_2, v_2\lambda_1\}, \]
where $v_i$, $\delta_i^+$, $\delta_i^-$, $\kappa_i$ and $\lambda_i$ are the order, the minimum out-degree, the minimum in-degree, the connectivity and the edge-connectivity of $D_i$, respectively.
Some Remarks

- The second method (by estimating the size of minimum separating set) is more powerful.
- Can we apply the Menger Method to prove the vertex case and the case of digraphs?
References

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Thank you!