Uniform Star-factors of Graphs with Girth Three

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Definitions and Notations

A *leaf* is a vertex of degree one and a *stem* is a vertex which has at least one leaf as a neighbor.

The length of a shortest cycle in graph G is called the *girth* of G.

A *star* is a tree isomorphic to $K_{1,n}$ for some $n \geq 1$. The vertex of degree n is called the *center* of the star.

A *star-factor* of a graph G is a spanning subgraph of G such that each component of which is a star.

Definitions and Notations

Theorem (*Amahashi and Kano, 82*) *Let G be a graph and n be an integer greater than or equal to 2. Then G has a $\{K_{1,1}, K_{1,2}, \cdots, K_{1,n}\}$-factor if and only if $i(G-S) \leq n|S|$ for every $S \subset V(G)$.*

Definitions and Notations

An *edge-weighting* of a graph G is a function

$$w : E(G) \longrightarrow \mathbb{N}$$

If H is a subgraph of G then $w(H)$, the weight of H under w, is

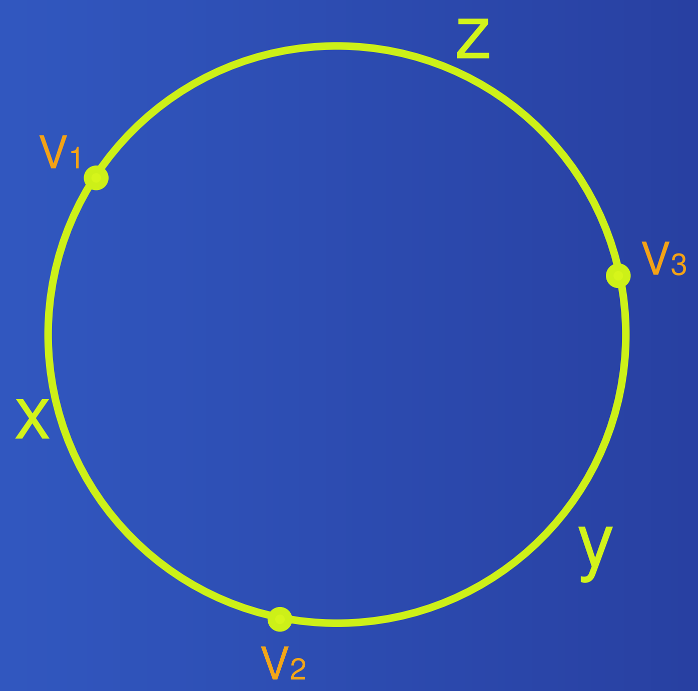
$$w(H) = \Sigma_{e \in E(H)} w(e)$$

Definitions and Notations

Question *For a given graph $G = (V, E)$ is there an edge-weighting w of G such that every star factor of G has the same weight under w?*

Examples

1. Uniform solution

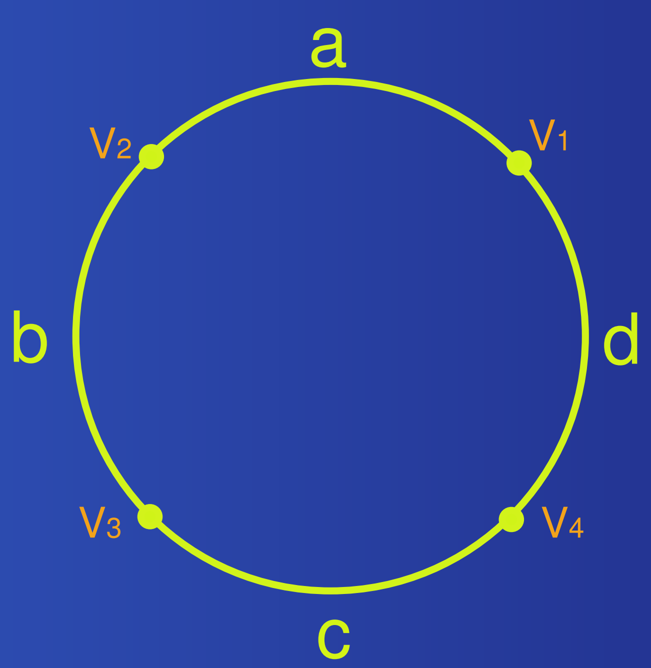


$$x+y=y+z=z+x$$

$$x=y=z$$

Examples

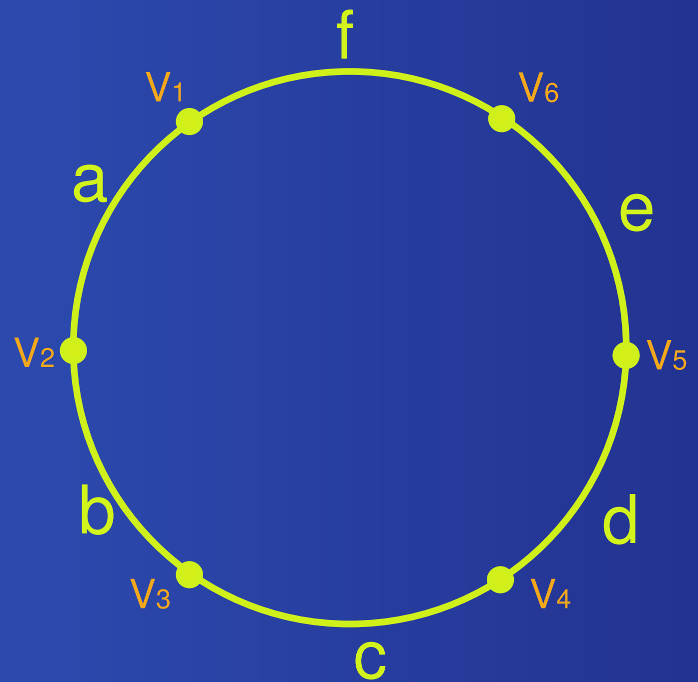
2. Infinite solutions



a+c=b+d

Examples

3. No solution



a+c+e=b+d+f=a+b+d+e=b+c+e+f=c+d+f+a=k

a+b+c+d+e+f=2k

2(a+b+c+d+e+f)=3k

Definitions and Notations

$w : w(e) = l$, for each $e \in E(G)$ and $l \neq 0$.

Note: Every star-factor of G has the same weight if and only if all star-factors have the same number of edges. In fact, the same number of components.

Definitions and Notations

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Note: Every star-factor of G has the same weight if and only if all star-factors have the same number of edges. In fact, the same number of components.

We denote by $\mathscr{U}$ the set of all graphs G such that every star-factor of G has the same number of edges.

Girth at least five

Theorem (*Hartnell and Rall, 05*) *Let G be a connected graph of girth at least five and minimum degree at least two. Then $G \in \mathscr{U}$ if and only if G is a 5-cycle or a 7-cycle.*

Girth at least five

Theorem (*Hartnell and Rall, 05*) G *is a connected graph of girth at least five and contains at least one leaf. Then* $G \in \mathscr{U}$ *if and only if each component of the graph* G *obtained by removing all the leaves and stems from* G *is:*

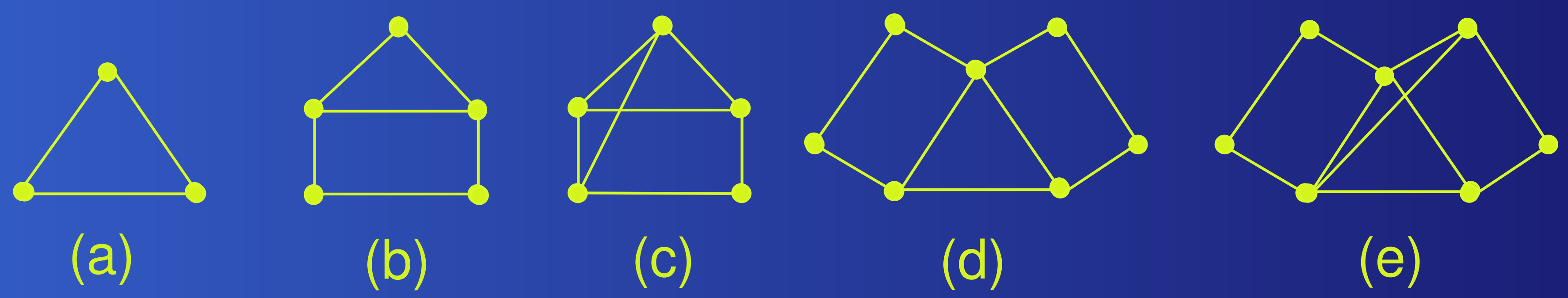
$1.$ *A* 5*-cycle in which at most two vertices have degree more than two in* G *such that if there are two such vertices they are nonadjacent on the* 5*-cycle;*

$2.$ *A star* $K_{1,m}$ *for some* $m \geq 2$ *such that the center of this star has degree* m *in* G*;*

$3.$ *An isolated vertex.*

Girth three

Theorem (*Wu and Yu, 06*) *Let G be a connected graph of girth three and minimum degree at least two. Then $G \in \mathscr{U}$ if and only if G is one of the following five graphs shown bellow.*



Girth three

Lemma *If F is a subset of $E(G)$ such that $G - F$ has no isolated vertices and $G - F$ is not in $\mathscr{U}$, then G is not in $\mathscr{U}$.*

Lemma *Let G be a graph in $\mathscr{U}$ with an induced triangle. If exactly one of the vertices on this triangle has degree at least three, then all of its neighbors that don't belong to this triangle must be stems.*

Idea of proof:

Assume G contains a triangle $C_3 = v_1v_2v_3$ with at least two vertices of degree at least three by lemma above.

Case $1)$ $d_G(v_3) = 2$, $d_G(v_1) \geq 3$, $d_G(v_2) \geq 3$.

Case $2)$ $d_G(v_3) \geq 3$, $d_G(v_1) \geq 3$, $d_G(v_2) \geq 3$.

Girth three

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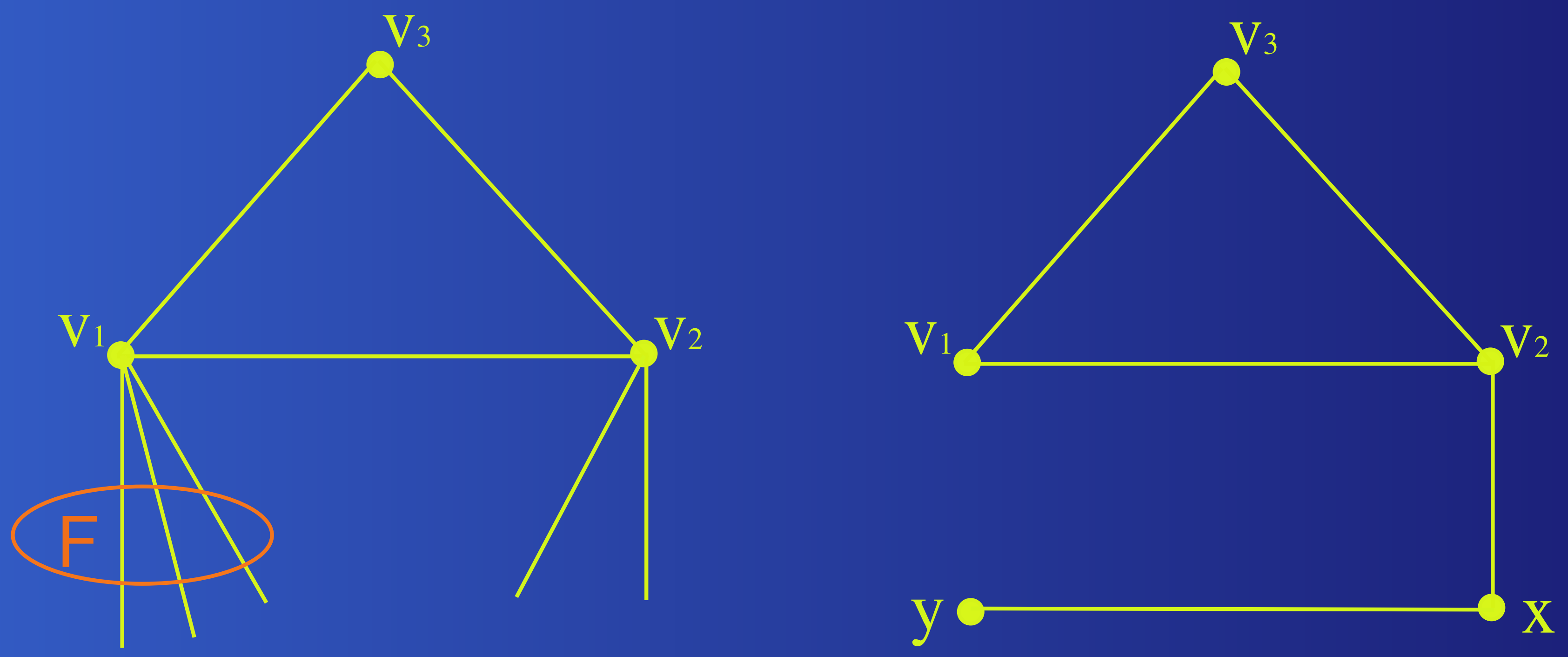

Girth three

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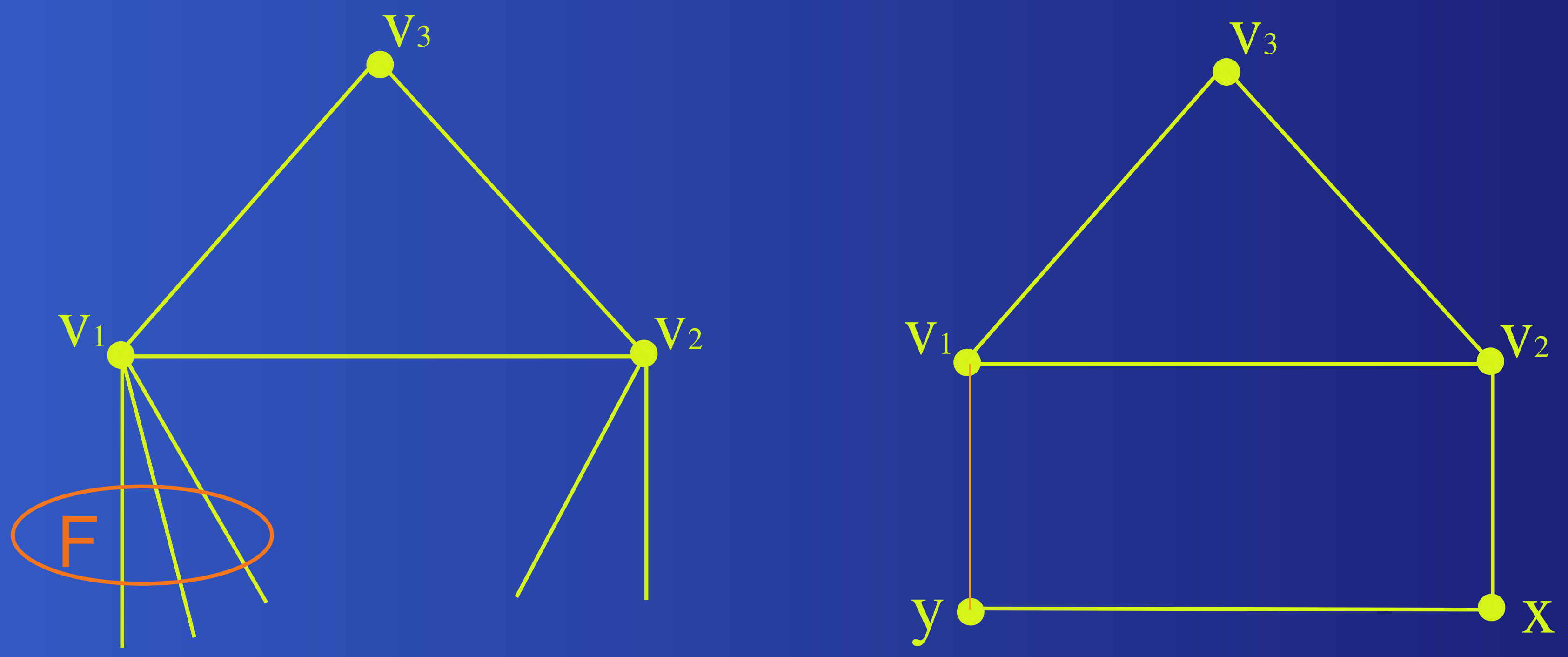

Girth three

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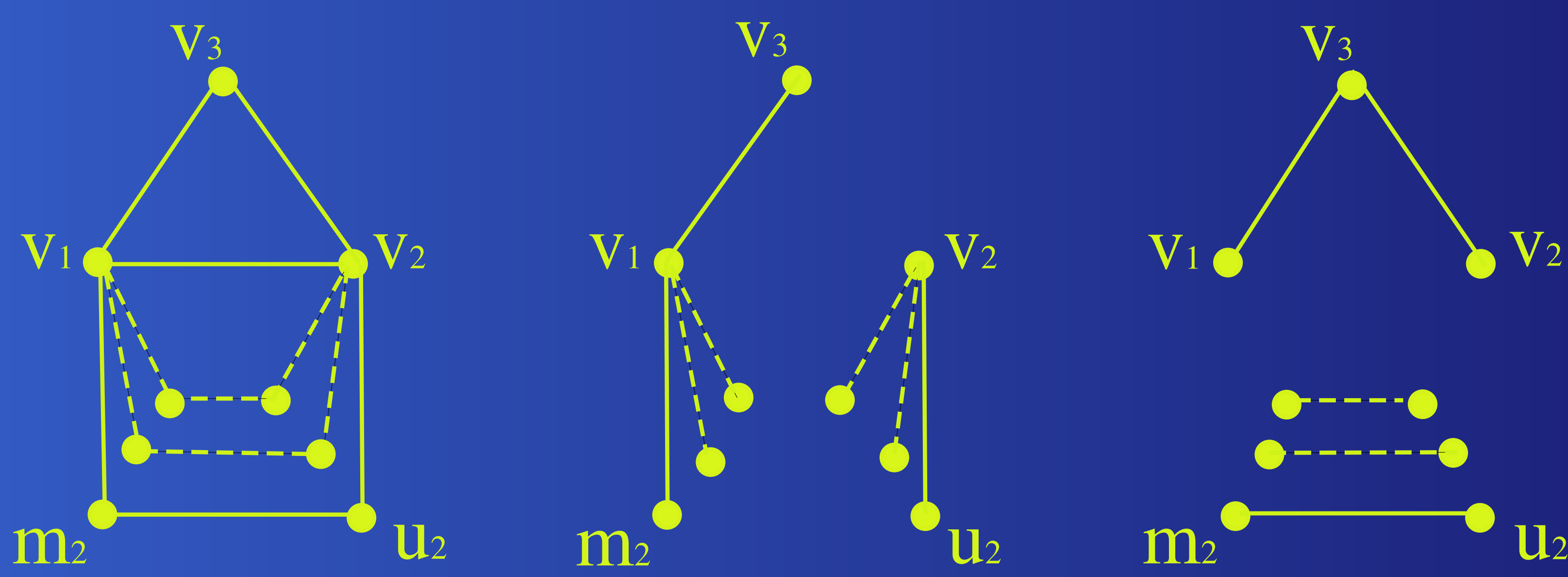


(a)

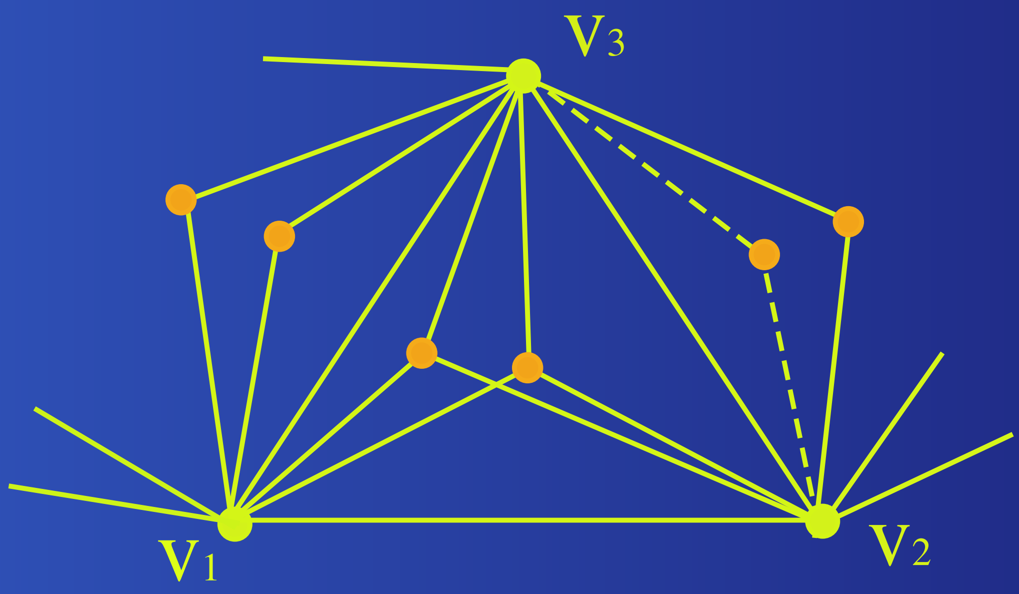
(b)

(c)

Girth three

Case $2)$ $d_G(v_3) \geq 3$, $d_G(v_1) \geq 3$, $d_G(v_2) \geq 3$.

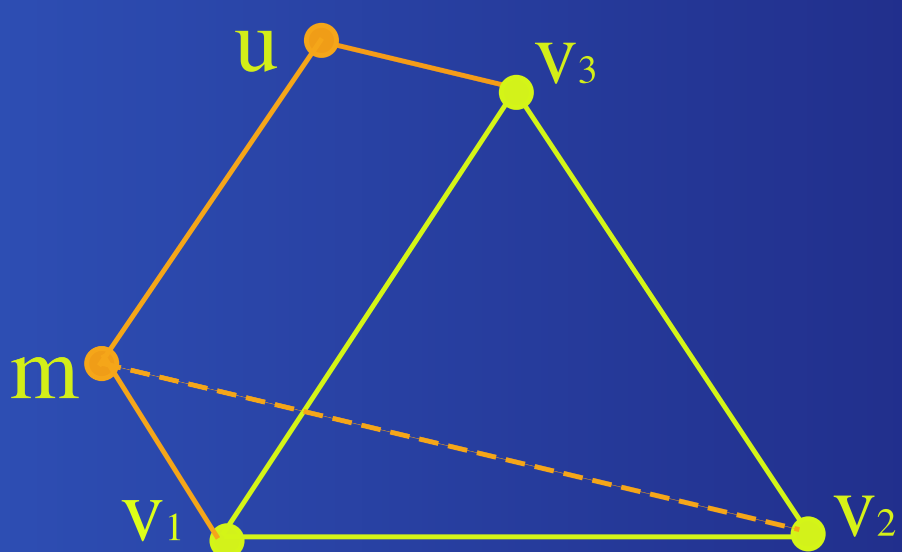
Claim 1: $\{x \in V(G) \mid N_G(x) \subseteq \{v_1, v_2, v_3\}\} = \emptyset$.



Girth three

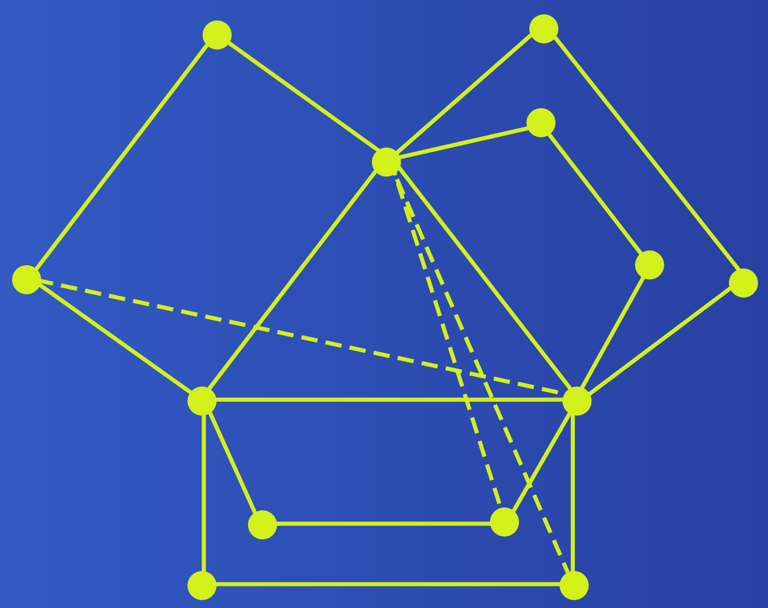
Case 2) $d_G(v_3) \geq 3$, $d_G(v_1) \geq 3$, $d_G(v_2) \geq 3$.

Claim 2: $N_G(u) - \{m, v_3\} \subseteq \{v_1, v_2\}$ and $N_G(u) \cap N_G(m) = \emptyset$.

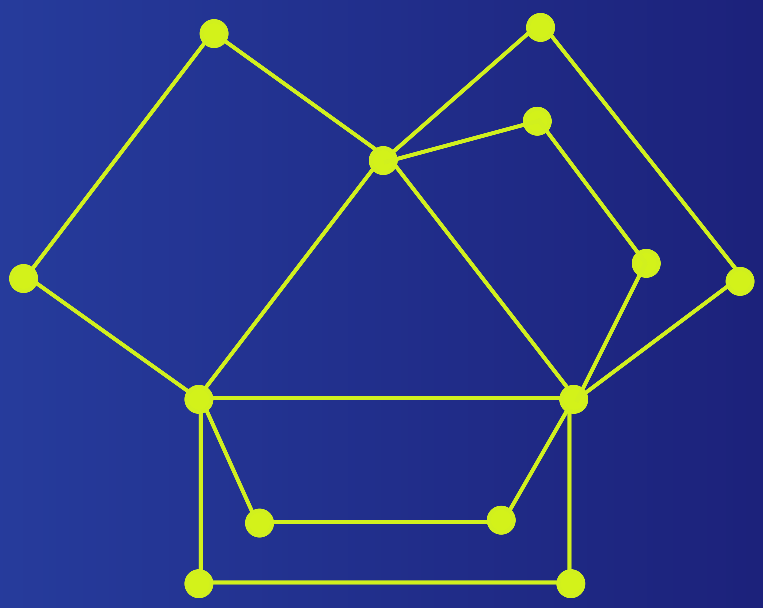


Girth three

Case $2)$ $d_G(v_3) \geq 3$, $d_G(v_1) \geq 3$, $d_G(v_2) \geq 3$.



(G)

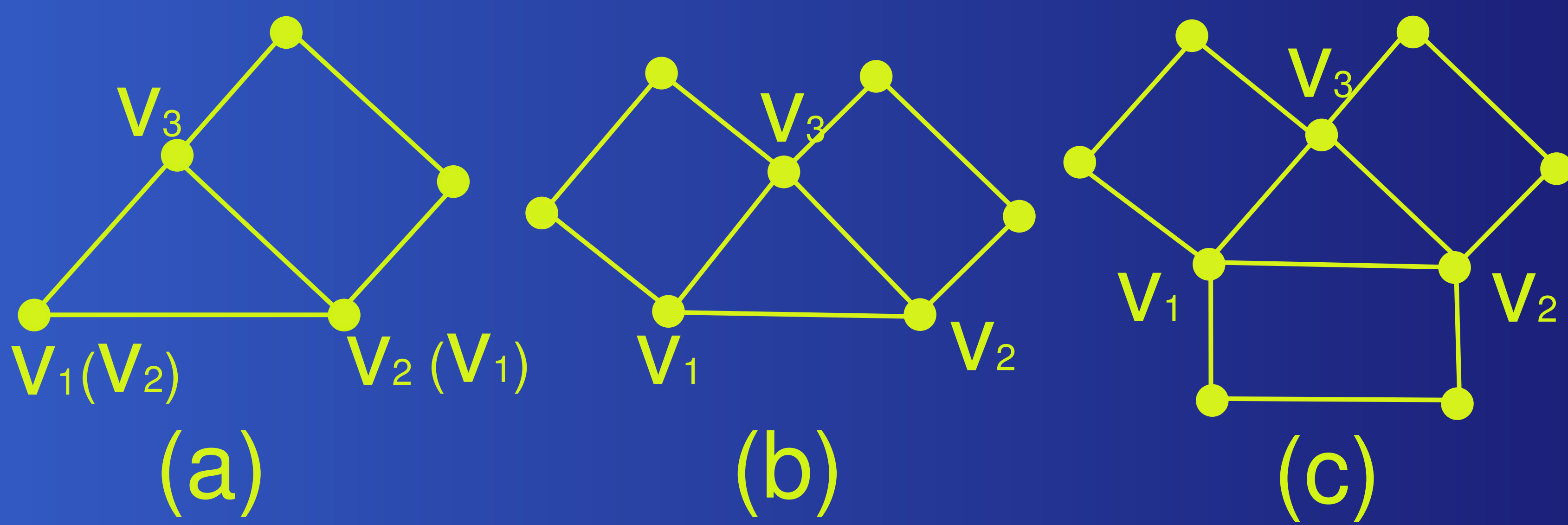


(G')

Girth three

Case $2)$ $d_G(v_3) \geq 3$, $d_G(v_1) \geq 3$, $d_G(v_2) \geq 3$.

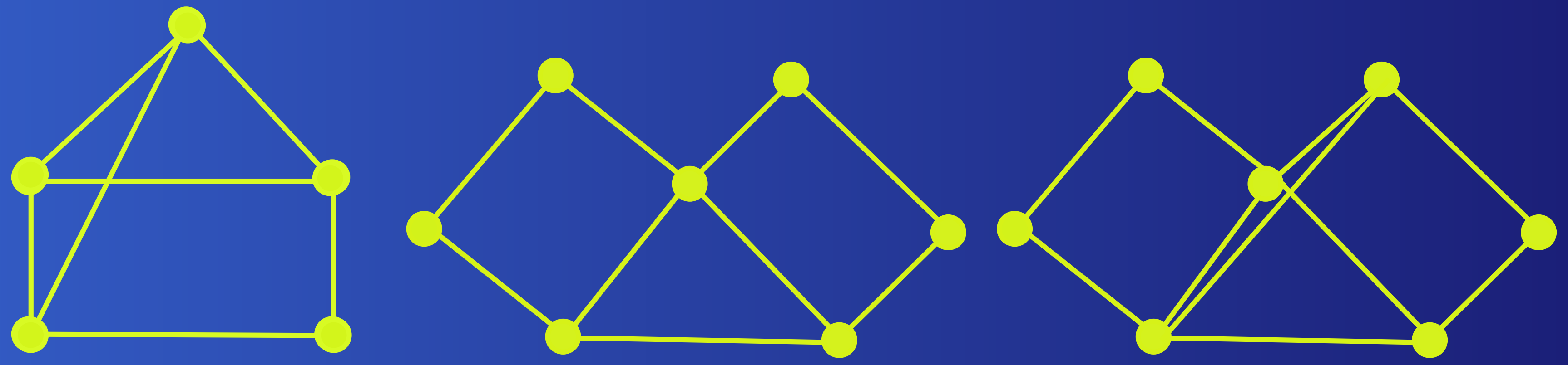
Claim 3: For each edge of the triangle $\Delta v_1 v_2 v_3$, there exits at most one 4-cycle in G' containing it.



Girth three

Case $2)$ $d_G(v_3) \geq 3$, $d_G(v_1) \geq 3$, $d_G(v_2) \geq 3$.

We add the edges back by the principle of claim 2.



THANK YOU!