# Removable ears of 1-extendable graphs (or matching covered graphs)

Zhai Shaohui

School of Mathematical Sciences
Xiamen University

2006. 6. 10

# 1 Ear decomposition

Let $G$ be a graph. A single ear of $G$ is a path $P$ of odd length in $G$ whose internal vertices (if any) all have degree two in $G$ and whose end vertices have degree greater than 2 in $G$. A double ear of $G$ is a pair $(P_1, P_2)$, where $P_1$ and $P_2$ are two vertex-disjoint single ears of $G$.

An ear decomposition of a 1-extendable graph $G$ is a sequence $G_1 \subset G_2 \subset \cdots \subset G_r = G$ of 1-extendable subgraph of $G$ where

(i) $G_1 = K_2$;

(ii) for $2 \leq i \leq r$, $G_{i-1} = G_i - R_i$, where $R_i$ is an ear (single or double) of $G_i$.

The following fundamental theorem was established by Lovász and Plummer [*MATCHING THEORY, 1986*].

**Theorem 1.1.** (*The two-ear Theorem* ) Every 1-extendable graph has an ear decomposition.

There are various proofs of this theorem. The original proof due to Lovász and Plummer is somewhat involved. Later on, Litter and Rendl [*J.Austral. Math. Soc,(Series A), 46 (1989)*] gave a simpler proof. Recently, another simple proof was given by Szigeti [*J. Combin. Theory Ser. B*, 74 (1998)].

# 2 Removable ear

Let $G$ be a 1-extendable graph and $R$ an ear (single or double) of $G$. Denote by $G-R$ the graph obtained from $G$ by deleting the edges and internal vertices of the constituent paths of $R$. A single ear $R$ of $G$ is removable if the graph $G - R$ is 1-extendable. A removable single ear of length one is called a removable edge. The notion of a removable double ear $R = (P_1, P_2)$ in $G$ is similarly defined, where neither $P_1$ nor $P_2$ is a removable single ear in $G$. A removable ear in $G$ is either a single or double ear which is removable. If $G$ a is brick, then each removable edge of $G$ corresponds to a removable single ear, and each removable doubleton corresponds to a removable double ear.

Lovász [*Combinatorica,* 3 (1983)] proved the following result.

**Theorem 2.1.** (*The removable edge Theorem*) Every brick different from $K_4$ and $\overline{C_6}$ has a removable edge.

Carvalho, Lucchesi and Murty [*Combinatorica* 19 (1999)] gave the following two theorems.

**Theorem 2.2.** Let $G$ be a brick distinct from $K_4$ and $\overline{C_6}$, then it has at least $\Delta(G) - 2$ removable edges.

Theorem 2.2 is a generalization of Theorem 2.1.

**Theorem 2.3.** Let $G$ be a 1-extendable graph different from $K_2$ and $C_{2n}$, then it has at least $\Delta(G)$ edge-disjoint removable ears.

Theorem 2.3 implies that any 1-extendable graph $G$ has at least $(\Delta(G))!$ distinct ear decompositions and thus is a generalization of the fundamental theorem of Lovász and Plummer on the existence of ear decompositions.

Pierre Hansen, Fuji Zhang and Maolin Zheng [*Discrete Mathematics* 176 (1997)] gave the following theorem.

**Theorem 2.4.** For an elementary bipartite graph $G$ which is neither a cycle nor $K_2$, $\mu(G) + 1 \leq er(G) \leq 3(\mu(G) - 1)$.

# 3 Our results

**Lemma 3.1.** Let $G$ be a graph, then $E(G)$ can be partitioned as at least $\chi'(G)$ edge-disjoint matchings which cover all edges of $G$.

**Lemma 3.2.** Let $G$ be a brick, then it has at least $\chi'(G)$ edge-disjoint removable ears.

**Lemma 3.3.** If $G$ is a $n$-extendable graph with $2n \leq |V(G)| - 2$ , then for any edge $e \in E(G)$, $G - e$ is $(n-1)$-extendable.

**Theorem 3.4.** Let $G$ be a 1-extendable graph different from $K_2$ and $C_{2n}$, then it has at least $\chi'(G)$ edge-disjoint removable ears.

**Theorem 3.5.** Let $G$ be a brick distinct from $K_4$ and $\overline{C_6}$, then it has at least $\chi'(G) - 2$ removable edges.

## 4 Tight cut decomposition

Lovász [*J. Combin. Theory Ser. B,* 43 (1987)] proved the following deep theorems which shows the important of the two types of tight cuts.

**Theorem 4.1.** If a 1-extendable graph $G$ has a nontrivial tight cut, then it has either a barrier cut or a 2-separation cut.

**Theorem 4.2.** A 1-extendable graph has no nontrivial tight cuts if and only if it is either a brick or a brace.

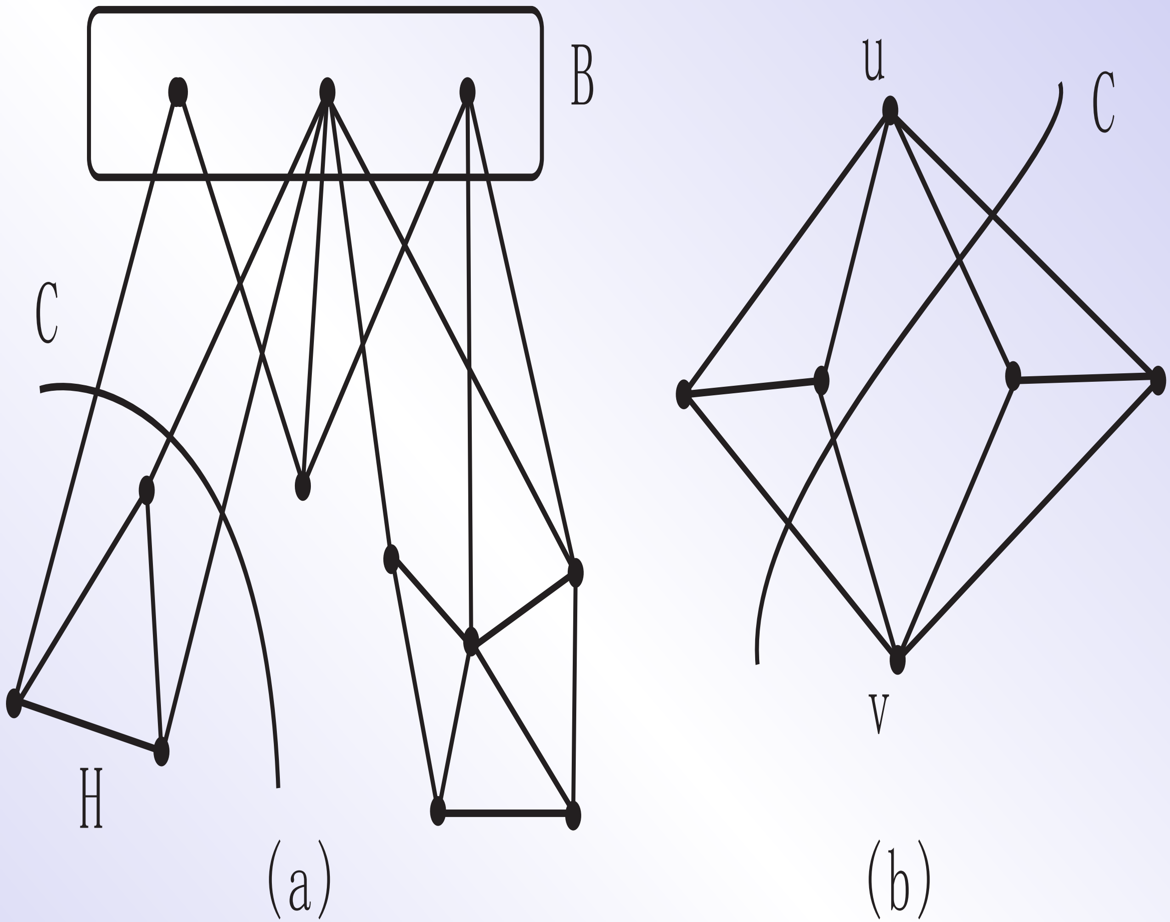


Figure 1: Barrier cuts and 2−separation cuts

1

# 5 A dependence relation

Let $G$ be a 1-extendable graph, and let $e$ and $f$ be any two edges of $G$. Then $e$ depends on $f$ if every perfect matching that contains $e$ also contains $f$. Let $e \Rightarrow f$ indicate that $e$ depends on $f$. Clearly, $\Rightarrow$ is reflexive and transitive. It is convenient to visualize $\Rightarrow$ in terms of the digraph it defines on the set of edges of $G$. Carvalho [Ph.D. thesis (1997)] at the first time introduced the definition.

Two edges $e$ and $f$ are mutually dependent (simply, $e \Leftrightarrow f$) if $e \Rightarrow f$ and $f \Rightarrow e$. Obviously, $\Leftrightarrow$ is an equivalence relation on $E(G)$. Let $D(G)$ be the digraph, whose vertex set consists of all equivalent classes on $E(G)$ and, $(X, Y)$ is an arc in $D(G)$ if and only if any perfect matching of $G$ that contains $X$ also contains $Y$. Clearly, $D(G)$ is acyclic. The sources in this digraph are called minimal classes. Let $e$ be an edge of $G$ and $[e]$ the equivalent class of $E(G)$ containing $e$. Given an edge $e$ of $G$, consider the subdigraph of $D(G)$ induced by the set of all the equivalence classes $[f]$ such that $f \Rightarrow e$. Then, a minimal class in this subdigraph of $D(G)$ is clearly a minimal class in $D(G)$ itself; it is said to be induced by $e$.

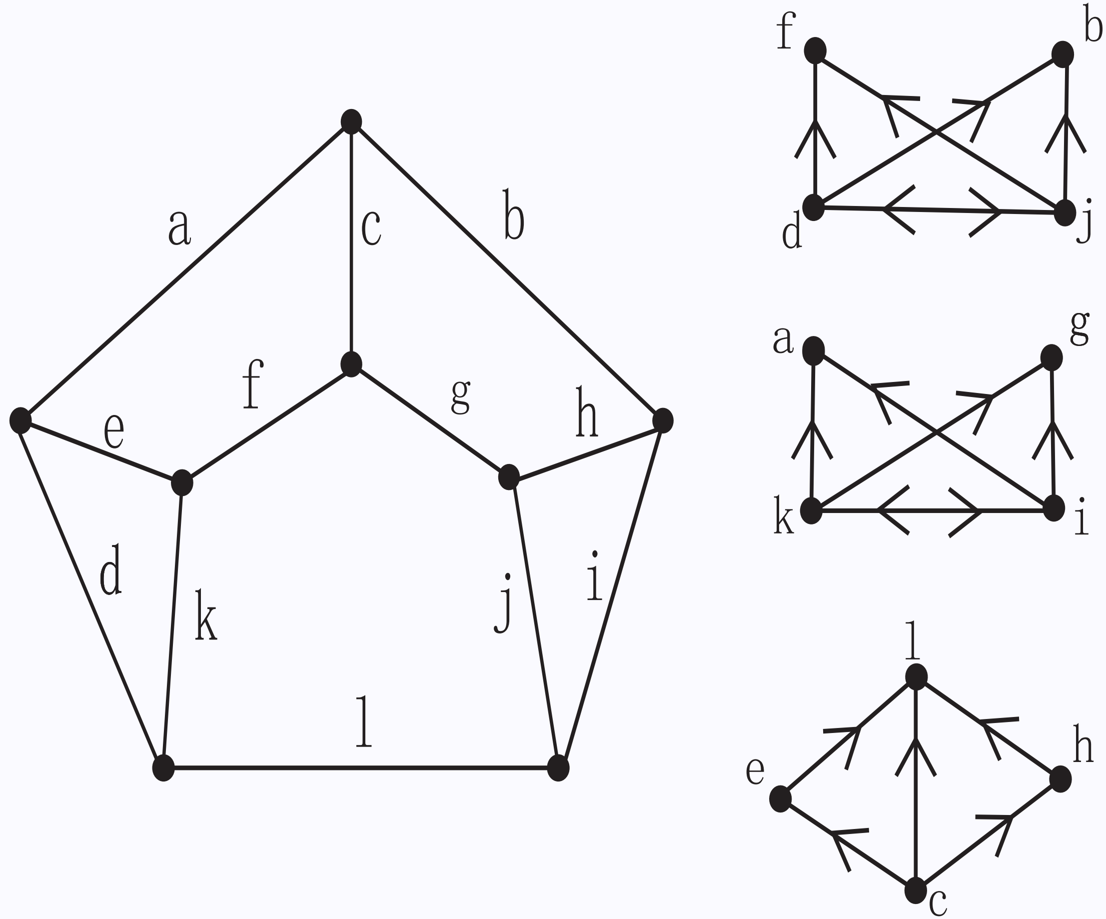


Figure 1: A 1-extendable graph $G$ and dependence relation on $E(G)$

1

# 谢谢大家!

Thank you very much!