Some results on the existence of factors, connected factors and fractional factors

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1. Introduction

Let $G$ be a graph with vertex set $V(G)$ and let $f$ be a nonnegative integer-valued function defined on $V(G)$. A spanning subgraph $F$ of $G$ is called an $(g, f)$-factor if $g(x) \leq d_F(x) \leq f(x)$ for every $x \in V(F)$. A connected $(g, f)$-factor is a connected one.
A fractional \((g,f)\)-indicator function is a function \(h\) that assigns to each edge of a graph \(G\) a number \(h(e)\) in \([0,1]\) so that for each vertex \(x\) we have \(g(x) \leq d^h_G(x) \leq f(x)\), where \(d^h_G(x) = \sum_{x \in E_x} h(e)\) is the fractional degree of \(x\) in \(G\) with \(E_x = \{e : e = xy \in E(G)\}\).
When \( g(x) = f(x) \) for every \( x \in V(G) \), a fractional \((g,f)\)-indicator function is called a fractional \( f \)-indicator function. Let \( h \) be a fractional \( f \)-indicator function of a graph \( G \). Set \( E_h = \{ e : e \in E(G) \text{ and } h(e) \neq 0 \} \). If \( G_h \) is a spanning subgraph of \( G \) such that \( E(G_h) = E_h \), then \( G_h \) is called a fractional \( f \)-factor of \( G \).
2. Main results

There is a well-known necessary and sufficient condition for a graph $G$ to have an $f$-factor.

**Theorem 1.** (W. T. Tutte, Can. T. Math, 1952) A graph $G$ has an $f$-factor if and only if

$$\delta(S, T) = f(S) + d_{G-S}(T) - f(T) - h(S, T) \geq 0$$

for any disjoint subsets $S$ and $T$ of $V(G)$, where $h(S, T)$ denotes the number of odd components $C$ of $G - (S \cup T)$. 
A similar result for the existence of fractional factors was given by Liu and Zhang.

**Theorem 2.** (Liu and Zhang, Acta Math. Scientia, 2001) A graph $G$ has an fractional $f$-factor if and only if for every subset $S$ of $V(G)$

$$\gamma(S,T) = f(S) + \sum_{x \in T} d_{G-S}(x) - f(T) \geq 0$$

where $T = \{ x : x \in V(G) - S \text{ and } d_{G-S}(x) \leq f(x) \}$.
Theorem 3. (Cai and Liu, Ars, 2005) Let $G$ be a connected $K_{1,n}$-free graph and let $f$ be a nonnegative integer-valued function defined on $V(G)$ such that $1 \leq n-1 \leq a \leq f(x) \leq b$ for every $x \in V(G)$. If $f(V(G))$ is even, $\delta(G) \geq b + n - 1$, and $\alpha(G) \leq \frac{4a(\delta - b - n + 1)}{(b+1)^2(n-1)}$, then $G$ has an $f$-factor.
We conjecture that the similar result can be obtained in general graphs.

**Conjecture 1.** (Cai and Liu, Ars, 2005) Let $G$ be a connected graph and let $f$ be a nonnegative integer-valued function defined on $V(G)$ such that $a \leq f(x) \leq b$ for every $x \in V(G)$. If $\delta(G) \geq b$, $f(V(G))$ is even and $\alpha(G) \leq \frac{4a(\delta-b)}{(b+1)^2}$, then $G$ has an $f$-factor.
Theorem 4. (Cai and Liu, 2006) Let $G$ be a connected graph of order $n$ and let $f$ be a nonnegative integer-valued function defined on $V(G)$ such that $1 \leq a \leq f(x) \leq b$ for every $x \in V(G)$. Where $n \geq (a+b)$. If $f(V(G))$ is even, $\delta(G) \geq bn/(a+b)$, and $\alpha(G) \leq \frac{4a(\delta-b)}{(b+1)^2}$, then $G$ has an $f$-factor.

when $n = a + b$, this result implies that conjecture 1 is true.
In addition, another result implies that the fractional analogue of Conjecture 1 is true.

**Theorem 5.** (Cai and Liu, Ars, 2006) Let $G$ be a connected graph and let $f$ be a nonnegative integer-valued function defined on $V(G)$ such that $0 \leq a \leq f(x) \leq b$ for every $x \in V(G)$. If $\delta(G) \geq b$, and $\alpha(G) \leq \frac{4a(\delta-b)}{(b+1)^2}$, then $G$ has an fractional $f$-factor.
There is some results for the existence of Hamiltonian factors.

**Theorem 6.** (Cai and Liu, JAMC, 2006) Let $G$ be a connected graph of order $n$, $a$ and $b$ be integers such that $4 \leq a \leq b$. Let $g$ and $f$ be positive integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$ for every $x \in V(G)$. Suppose that $n \geq \frac{(a+b-5)^2}{a-2}$. If $\text{bind}(G) \geq \frac{(a+b-5)(n-1)}{(a-2)n-3(a+b-5)}$, and for any nonempty independent subset $X$ of $V(G)$, $|N_G(X)| \geq \frac{(b-3)n+(2a+2b-9)}{a+b-5}$, then $G$ has a Hamiltonian $(g, f)$-factor.
Theorem 7. (Cai, Liu and Hou, 2006) Let $G$ be a connected graph of order $n$, $a$ and $b$ be integers such that $3 \leq a \leq b$ and $b \geq 4$. Let $f$ be a positive integer-valued function defined on $V(G)$ such that $a \leq f(x) \leq b$ for every $x \in V(G)$. Suppose that $n \geq \frac{(a+b-4)^2}{a-2}$ and $\sum_{x \in V(G)} f(x) \equiv 0 \pmod{2}$. If $bind(G) \geq \frac{(3a+b-8)n-3(a+b)+16}{(a-2)n-(a+b-6)}$, then $G$ has a Hamiltonian $f$-factor.
For the stability number, degree condition and the existence of connected \([k, k+1]\)-factor, there is the following result.

**Theorem 8.** (Cai, Liu and Hou, 2006) Let \(G\) be a 2-connected graph of order \(n\) and let \(k \geq 2\) be a positive integer such that \(n \geq 3k - 1\) and \(kn\) is even. If \(\alpha(G) \leq \frac{4k(\delta-k)}{(k+1)^2}\), \(\delta(G) \geq \frac{n+1}{3}\), and for each pair of nonadjacent vertices \(u, v\) in \(G\)

\[
\max\{d(u), d(v)\} \geq \frac{n - k}{2}.
\]

Then \(G\) has a connected \([k, k+1]\)-factor.
The following result shows the relationship between binding number and the existence of connected \([k, k+1]\)-factor.

**Theorem 9.** (Cai, Liu and Hou, 2006) Let \(G\) be a connected graph of order \(n\), and let \(k\) be integers such that \(k \geq 2\) and \(kn\) is even. If \(n \geq 4k - 2\), \(\text{bind}(G) \geq \frac{(2k-1)(n-1)}{(k-1)n+2k-1}\). Then \(G\) has a connected \([k, k+1]\)-factor.
3. Related problems and conjectures for further research.

(1) (The existence of fractional factors)

**Problem 1.** (Liu and Zhang, Advances in Math, 2006) *Find the relationship between binding number and fractional k-extendable of a graph.*

**Problem 2.** (Cai and Liu, 2006) *Find the relationship between the toughness and the existence of fractional f-factors of a graph.*

Conjecture 2. (Liu and Zhang, Advances in Math, 2006) *Let $G$ be a graph. If $t(G) \geq k$, then $G$ has a connected fractional $k$-factor.*
(2) (The existence of factors)

**Problem 4.** (Cai and Liu, 2006) *Find the relationship between fan-type condition and the existence of $f$-factors in graphs.*

**Problem 5.** (Cai and Liu, 2006) *Find the relationship between the isolated toughness condition and the existence of $(g, f)$-factors of graphs.*

**Conjecture 1.** (Cai and Liu, 2006).
Conjecture 3. (Liu and Zhang, Advances in Math, 2001) Let $G$ be a graph. If $t(G) \geq k$, then $G$ has a connected $[k, k + 1]$-factor.

problem 6. (Cai and Liu, 2006) Find the relationship between fan-type condition and the existence of connected $(g, f)$-factors of graphs.
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Thank You!