Some results on the existence of factors , connected factors and fractional factors

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1. Introduction

Let G be a graph with vertex set V(G) and let f be a nonnegative integer-valued function defined on V(G). A spanning subgraph F of G is called an (g, f)-factor if $g(x) \leq d_F(x) \leq f(x)$ for every $x \in V(F)$. A connected (g, f)-factor is a connected one.

A fractional (g, f)-indicator function is a function h that assigns to each edge of a graph G a number $h(e)$ in [0,1] so that for each vertex x we have $g(x) \leq d_G^h(x) \leq f(x)$, where $d_G^h(x) = \sum_{x \in E_x} h(e)$ is the fractional degree of x in G with $E_x = \{e : e = xy \in E(G)\}$.

When $g(x) = f(x)$ for every $x \in V(G)$, a fractional (g,f)-indicator function is called a fractional f-indicator function. Let h be a fractional f-indicator function of a graph G. Set $E_h = \{e : e \in E(G) \quad and \quad h(e) \neq 0\}$. If G_h is a spanning subgraph of G such that $E(G_h) = E_h$, then G_h is called a fractional f-factor of G

2. Main results

There is a well-known necessary and sufficient condition for a graph G to have an f-$factor$.

Theorem 1. (W. T. Tutte, Can. T. Math, 1952) *A graph G has an f-$factor$ if and only if*

$$\delta(S,T) = f(S) + d_{G-S}(T) - f(T) - h(S,T) \geq 0$$

for any disjoint subsets S and T of V(G), where $h(S,T)$ denotes the number of odd components C of $G-(S\bigcup T)$.

A similar result for the existence of fractional factors was given by Liu and Zhang.

Theorem 2. (Liu and Zhang, Acta Math. Scientia, 2001) *A graph G has an fractional f-$factor$ if and only if for every subset S of V(G)*

$$\gamma(S,T) = f(S) + \sum_{x \in T} d_{G-S}(x) - f(T) \geq 0$$

where $T = \{x : x \in V(G) - S$ *and* $d_{G-S}(x) \leq f(x)\}$.

Theorem 3. (Cai and Liu, Ars, 2005) *Let G be a connected $K_{1,n}$-free graph and let f be a nonnegative integer- valued function defined on V(G) such that* $1 \leq n-1 \leq a \leq f(x) \leq b$ *for every* $x \in V(G)$*. If $f(V(G))$ is even,* $\delta(G) \geq b+n-1$, *and* $\alpha(G) \leq \frac{4a(\delta-b-n+1)}{(b+1)^2(n-1)}$, *then G has an f-factor*

We conjecture that the similar result can be obtained in general graphs.

Conjecture 1.(Cai and Liu, Ars, 2005) *Let G be a connected graph and let f be a nonnegative integer-valued function defined on V(G) such that $a \leq f(x) \leq b$ for every $x \in V(G)$. If $\delta(G) \geq b$, f(V(G)) is even and $\alpha(G) \leq \frac{4a(\delta-b)}{(b+1)^2}$, then G has an f-$factor$.*

Theorem 4. (Cai and Liu, 2006) *Let G be a connected graph of order n and let f be a nonnegative integer- valued function defined on V(G) such that $1 \leq a \leq f(x) \leq b$ for every $x \in V(G)$. Where $n \geq (a+b)$. If $f(V(G))$ is even, $\delta(G) \geq bn/(a+b)$, and $\alpha(G) \leq \frac{4a(\delta-b)}{(b+1)^2}$, then G has an f-factor.*

when $n = a + b$, this result implies that conjecture 1 is true.

In addition, another result implies that the fractional analogue of Conjecture 1 is true.

Theorem 5. (Cai and Liu, Ars, 2006)*Let G be a connected graph and let f be a nonnegative integer-valued function defined on V(G) such that* $0 \leq a \leq f(x) \leq b$ *for every* $x \in V(G)$. *If* $\delta(G) \geq b$, *and* $\alpha(G) \leq \frac{4a(\delta-b)}{(b+1)^2}$, *then G has an fractional* f*-factor*.

There is some results for the existence of Hamiltonian factors.

Theorem 6. (Cai and Liu, JAMC, 2006) *Let G be a connected graph of order n, a and b be integers such that $4 \leq a \leq b$. Let g and f be positive integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$ for every $x \in V(G)$. Suppose that $n \geq \frac{(a+b-5)^2}{a-2}$. If $bind(G) \geq \frac{(a+b-5)(n-1)}{(a-2)n-3(a+b-5)}$, and for any nonempty independent subset X of $V(G)$, $\mid N_G(X) \mid \geq \frac{(b-3)n+(2a+2b-9)|}{a+b-5}$ Then G has a Hamiltonian (g, f)-factor.*

Theorem 7.(Cai, Liu and Hou, 2006) *Let G be a connected graph of order n, a and b be integers such that* $3 \leq a \leq b$ *and* $b \geq 4$. *Let f be a positive integer-valued function defined on V(G) such that* $a \leq f(x) \leq b$ *for every* $x \in V(G)$. *Suppose that* $n \geq \frac{(a+b-4)^2}{a-2}$ *and* $\sum\limits_{x \in V(G)} f(x) \equiv 0 (mod \quad 2)$. *If* $bind(G) \geq \frac{(3a+b-8)n-3(a+b)+16}{(a-2)n-(a+b-6)}$, *then G has a Hamiltonian f-factor.*

For the stability number, degree condition and the existence of connected [k, k+1]-factor, there is the following result.

Theorem 8. (Cai, Liu and Hou, 2006) *Let G be a 2-connected graph of order n and let $k \geq 2$ be a positive integer such that $n \geq 3k - 1$ and kn is even. If $\alpha(G) \leq \frac{4k(\delta - k)}{(k+1)^2}$, $\delta(G) \geq \frac{n+1}{3}$, and for each pair of nonadjacent vertices u,v in G*

$$max\{d(u), d(v)\} \geq \frac{n-k}{2}.$$

Then G has a connected $[k, k+1]$-factor.

The following result shows the relationship between binding number and the existence of connected [k, k+1]-factor.

Theorem 9. (Cai, Liu and Hou, 2006) *Let G be a connected graph of order n, and let* k *be integers such that* $k \geq 2$ *and* kn *is even. If* $n \geq 4k-2$, $bind(G) \geq \frac{(2k-1)(n-1)}{(k-1)n+2k-1}$*. Then G has a connected* $[k, k+1]$*-factor.*

3. Related problems and and conjectures for further research.

(1) (The existence of fractional factors)
Problem 1. (Liu and Zhang, Advances in Math, 2006) *Find the relationship between binding number and fractional k-extendable of a graph* .

Problem 2. (Cai and Liu, 2006) *Find the relationship between the toughness and the existence of fractional f-factors of a graph.*

Problem 3. (Liu and Zhang, Advances in Math, 2006) *Find the relationship between fractional factors and the factors in graphs.*

Conjecture 2. (Liu and Zhang, Advances in Math, 2006) *Let G be a graph. If* $t(G) \geq k$*, then G has a connected fractional k-factor.*

(2) (The existence of factors)

problem 4. (Cai and Liu, 2006) *Find the relationship between fan-type condition and the existence of f-factors in graphs.*

problem 5. (Cai and Liu, 2006) *Find the relationship between the isolated toughness condition and the existence of (g, f)-factors of graphs.*

Conjecture 1. (Cai and Liu, 2006).

(3) (The existence of connected factors)

Conjecture 3. (Liu and Zhang, Advances in Math, 2001) *Let G be a graph. If* $t(G) \geq k$*, then G has a connected* $[k, k+1]$*-factor.*

problem 6. (Cai and Liu, 2006) *Find the the relationship between fan-type condition and the existence of connected* (g, f)*-factors of graphs.*

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Thank You!